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EFFECT OF INDUSTRIAL EFFLUENTS ON THE GROWTH BEHAVIOUR OF WHEAT PLANTS

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ABSTRACT

Inhibition in different growth parameters of wheat plants caused by BHEL and IDPL effluents were investigated. After 18 weeks as compared to control, the percent reduction in shoot length, root length, number of leaves, number of seeds per inflores–cence in the plants treated with BHEL effluent were 58. 82 58. 85, 58. 83, 77. 77, 54. 14 and 49. 17 respectively, and under IDPL effluent effect the respective percent reduction stood at 22. 89, 19. 14, 35. 29, 44. 44, 26. 58 and 19 18.

Key words : Industrial effluents, physico-chemical, plant growth.

INTRODUCTION

Industrialisation is believed to cause inevitable problem of pollution of water, soil and air. Industrial effluents used for irrigation may contain heavy metals ond other organic and inorganic toxic materials which effect the gtowth of crop plants [4], [8]. The problem is of immense concern in Rishikesh and Hardwar where industrial and sewage drains are directly or indirectly being diverted to irrigate the crop field.

The present investigation deals with the effect of effluents of Bharat Heavy Electricals Limited (Hardwar) and Indian Drugs and Pharmaceuticals Limited (Rishikesh) on the growth behaviours fo wheat plants.

MATERIALS AND METHODS

The effluents were analysed for physico-chemical analysis a dopting the standard methods [12]. Effect of effluents on the wheat plants was studied in cemented pots in ten replicates. Thoroughly mixed garden soil was used and uniformly distributed in the pots. Different parameters of growth behaviour viz., shoot length, root length, leaf number, length of inflorescence and number of seeds per plants were measured initially at two weeks interval and then at four weeks interval. Observations were taken for 18 weeks from the Ist week of October, 1987. Tap water was used as control.

RESULTS AND DISCUSSION

The physico-chemical characteristics of both the industrial effluents are presented in Table 1. It was observed that effluent of BHEL was more turbid and contained larger amount of chloride, sulphate, nitrate, sodium, potassium and heavy metals as compared to IDPL effluent. The pH of IDPL effluent however, was slightly lower. It has been suggested [8] that such type of variation in the constituents of the effluents might be due to the different types of raw materials employed as well as the effluent treatment methods adopted by these two industries. As compared to control, the percent reduction in shoot length, root length and number

of leaves in the plants treated with BHEL effluents after 18 weeks were 58. 82, 58. 85 and 55. 88 respectively. while in the plants treated with IDPL effluents the respective percent values were 22. 89, 19. 14 and 35. 29. No reproductive structure was seen in 10th week in the plants treated with BHEL effluents, while 80 percent reduction were observed in plants treated with the effluents of IDPL. However, after 14 and 18 weeks the respective percent reduction in number of inflorescence and length of inflorescence in the plants treated with BHEL effluents were 77. 77 and 55. 14 respectively, while in the IDPL effluent treated Plants the respective reduction were 44. 44 and 26. 58 percent. The percent reduction in the number of seeds per inflorescence in the plants treated with the effluents of BHEL and IDPL were 49. 17 and 19. 18 respectively. It is evident that BHEL effluent delayed inflorescence formation.

It is also clear from the data (table 2) that growth of wheat plants was markedly effected by BHEL and IDPL effluents. Reduction in the growth of plants due to the effect of industrial effluents have also been reported by other workers [5], [9], [10], [11] and [13]. Such inhibition in the growth of plants are attributed to toxic effects of effluents such as presence of heavy metal, excess or deficit level of micronutrients and decomposition products as well as soil porosity and aeration [2], [4], [8] and [13]. The change in pH has also been noticed to effect the enzymatic processes governing the growth of the plants [1]. It has been concluded by Hasset et al. [3] that at high metal concentration selectivity of the cell memberane of the root of the plant reduces and thus allow more rapid entry of the metals into the plants. The growth inhibition could also be due to cytological disturbances [6]. It has also been noticed [7] that different effects observed in plants of polluted area are comparable to the features induced by chemical mutagens and conclusion has been drawn that these observations point to the genetic disturbances. The reduction in the growth parameters in the plants treated with IDPL effluent further revealed that the percent inhibition of plant growth is directly proportional to the concentration of the toxic elements in the effluent [13].

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Table 1. Physico-chemical characteristics of the industrial effluents.

S. No.	Factor	BHEL	IDPL
1.	Turbidity	162.53	139.87
2.	pH	7.53	6.70
3.	Total solids	170.25	120 05
4.	Chlorides as Cl.	126.82	129.45
5.	Sulphates as So_4	32.51	21.07
6.	Nitrate nitrogen	1.14	0.11
7.	Phosphates as Po_4	1.50	1.73
8.	Conductivity mohs/Cm	917.80	791.81
9.	Sodium as Na	66.56	50.30
10.	Potassium as K	21.8	10 9
11.	Copper as Cu	5.657	0.059
12.	Zinc as Zn	5.651	0 618
13.	Lead as Pb	3.14	0.91
14.	Manganese as Mn	20.833	10.697
15.	Iron as Fe	16.02	5.58
16.	Dissolved oxygen	2.42	2.80
17.	BOD	167.97	121.22
18.	COD	150.16	222.08

(An average value of investigated period in mg/1.)

Table 2. Effect of industrial effluents on the growth behaviour of wheat plants

Weeks	Effluents	Shoot length cm.	Root length cm.	No. leaves per plant	No. of Inflo-resence per plant	Length of inflorescence	No. of seed per inflore-seence
2	BHEL	8.70 ± 1.15	4.12 ± 0.50	1 ± 0.50	–	–	–
	IDPL	17.31 ± 1.25	11.[illegible] ± 0.25	2.5 ± 0.50	–	–	–
	Control	22.54 ± 1.1	13.59 ± 0.75	3.0 ± 0.50	–	–	–
6	BHEL	12.28 ± 1.25	5.50 ± 0.35	4.5 ± 0.20	–	–	–
	IDPL	32.70 ± 1.75	12.71 ± 0 65	6.5 ± 0.30	–	–	–
	Control	46.6 ± 2.25	15.50 ± 1.13	9.0 ± 1.0	–	–	–
10	BHEL	15.22 ± 1.50	7.23 ± 0.50	6.5 ± 0.50	–	–	–
	IDPL	38.50 ± 1.75	1[illegible].90 ± 0.35	11.0 ± 1.00	0.50 ± 0 50	–	–
	Control	50.10 ± 2.10	17.30 ± 1.60	17.5 ± 1.50	2.5 ± 0.50	–	–
14	BHEL	20.10 ± 0 55	7.30 ± 1.23	7.5 ± 0.50	1.0 ∓ 0.0	2.75 ± 0.55	–
	IDPL	40.40 ± 2.75	14.10 ± 0.50	11.0 ± 1.50	2.5 ± 0.50	4.80 ± 0.30	–
	Control	53.63 ± 2.30	17.50 ± 0.70	17.5 ± 1.50	4.5 ± 0.50	6.10 ± 0.42	–
18	BHEL	22.78 ± 0.50	7.20 ± 0.77	7.5 ± 0.50	1.0 ± 0.0	3.88 ± 0.35	15.63 ± 1.75
	IDPL	42.65 ± 2.68	14.15 ± 0.60	11.0 ± 1.50	2.5 ± 0.50	6.35 ± 0.53	14.85 ± 2.25
	Control	55.3[illegible] ± 2.40	17.50 ± 0.[illegible]0	17.0 ± 1.00	4.5 ± 0.50	8.65 ± 0.55	30.75 ± 2.15

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ASYMBIOTIC SEED GERMINATION OF *AERIDES MULTIFLORA* ROXB. AND *RHYNCHOSTYLIS RETUSA* BL. (ORCHIDACEAE).

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ABSTRACT

Seeds were obtained from green pods of *A. multiflora* and *A. retusa* and cultured on four different media i. e. Knudson 'C' modified medium, Mitra *et al.* medium, soil extract agar medium and soil extract coconut agar medium. The former two media were subjected with different combinations of growth regulators and organic substances. Mitra *et al.* medium was found more suitable for seed germination than others.

KEY WORDS : Protocorm, seed-germination, culture media, Orchid.

INTRODUCTION

The orchids are of great horticultural and medicinal value. A monographic account on their ecology, anatomy and medicinal properties have been published by Kaushik [12]. The researches on Indian orchids have been reviewed by Manilal and Kumar [17] and Chadha and Singh [6]. The

seeds of orchids are minute, microscopic, nonendospermic and are produced thousands and thousands in number within a single fruit capsule and are characterized with reduced embryo. It is surprising that only a few of them are fortunate to germinate in nature since they require a fungal association for germination [24]. Their viabilty can be tested by the technique given by Singh [26]. Several contributions are available in literature on seed germination, prominent among them are by Knudson [14], Noggle *et al.* [23], Hegarty [9] [10], Arditti [1] [2], Anderson [5] Harrison and Arditti [8], Stoutamire [28], Ernst [7], Arditti *et al.* [3] [4], Mitra [18], Murlidhar *et al.* [21], and Katiyar *et al.* [13]. It is believed that the fungus provide carbohydrates and other nutrients, by breaking down the fibrous seed coat, to initiate the growth [18]. Mature seeds from dehisced pods have low intensity for germination while seeds from green undehisced pods have higher intensity for germination and have been used by several investigators as by withner [35] [36], Ito [11], Nimoto and Sagawa [22], Sagawa [25], Valmayor and Sagawa [31], Teo and Teo [29], Vij *et al.* [32], Vij *et al.* [34]. Various culture media have been proposed by different authors for orchid seed germination [19, 20, 15, 30]. But none of these is found universal which can be used for either taxa of the orchidaceae. The present study was done to find out an appropriate recipe for the orchid-seed germination.

MATERIALS AND METHODS

The green pods of both the epiphytic orchids *A. multiflora* and *A. retusa* were collected from Dehradun in the month of November. Surface sterilized the green pods using 0.5% $HgCl_2$ (Mercuric chloride)

solution and washed with sterile water repeatedly. The pods were opened under aseptic conditions in the laminar flow using sterile surgical blade and the seeds were scooped out and inoculated with the help of sterile spatula in the culture tubes and 100 ml 'Erlenmeyers' conical flasks of Borosil. Cultures were subjected to 12 hours light period at 3500 lux followed by 12 hours dark period and kept on $25 \pm 2°C$ temperature.

Mitra *et al.* medium, Knudson' s 'C' modified medium, soil extract agar medium, and soil extract coconut agar medium, were the culture media. Sucrose was added to all media at the rate of 2%. Mitra *et al.* and Knudson 'C' media were supplemented with different concentrations of auxin, kinetin and organic substances and further experiments were done with Mitra *et al.* medium.

The compositions of aforesaid media are given in Table–1.

RESULTS

The orchid-seeds were considered germinated upon the emergence of embryo from its fibrous sheath. At first each medium was used as basal medium without any addition of auxin and growth regulators. The germination was highest in Mitra *et al.* medium and was found in decreasing order in Knudson's 'C' modified medium, S.E.C.A.M. and S.E.A.M. Response of S.E.A.M. was very poor to seed germination. Further experiments were conducted in Mitra *et al.* medium. Observations were recrded from time to time which are tabulated in Table 2

Table 1

Constituents	Mitra *et al.* medium (mg/l)	Modified Knudson's 'C'medium (mg/l)	Soil extract Coconut Agar medium	Soil extract Agar medium
Major elements				
Ca $(No_3)_2.4H_2o$	200	200		–
KNO_3	180	180	–	–
$(NH_4)_2SO_4$	100	100	–	–
NaH_2PO_4	150	–	–	–
K_2HPO_4	–	–	0.5gm/l	0.5gm/l
$MgSO_4.7H_2O$	250	250	–	–
Minor elements				
$MnSo_4.4H_2O$	–	0.075*	–	–
KI	0.03	0.80	–	–
$MnCl_2.4H_2O$	0.40	–	–	–
$ZnSO_4.7H_2O$	0.05	–	–	–
H_3BO_3	0.60	6.20	–	–
$CuSO_4.5H_2O$	0.05	0 025	–	–
$Na_2MoO_4.2H_2O$	0.05	0.250	–	–
$Co(NO_3)_2.6H_2O$	0.05	–	–	–
$ZnCl_2$	3.90	–	–	–
$CoCl_2$	–	0.025	–	–
$FeSO_4$	Na_2FeEDTA**	25.0	–	–
Vitamins				
Thiamin-HCl	0.30	0 30	–	–
Pyridoxir-HCl	0.30	0.30	–	–
Nicotinic acid	1.25	–	–	–
Riboflavin	0.05	0.30	–	–
Biotin	0.05	–	–	–
Folic acid	0.30	–	–	–
Sucrose	20 gm/l	20 gm/l	20 gm/l	20 gm/l
Agar	9.0 gm/l	9.0 gm/l	10 gm/l	10 gm/l
Soil extract***	–	–	100 ml/l	100 ml/l
Coconut water****	–	–	100 ml/l	-

* Separate solution of $MnSO_4.4H_2O$ was prepared by dissolving 1.5 mg in 100 ml of distilled water 5 ml/1 of this solution was used.

** Separate stock solution of Na_2 FeEDTA was prepared by dissolving 0.745 gm Na_2EDTA and 0.557 gm. $FeSO_4$ one by one in 100 ml. of distilled water. 3 ml/l of this solution was used.

*** To make soil extract, 1000 gm. of garden soil was taken and mixed it with 1000 ml. of water in a beaker and autoclaved for 30 minutes at 15 lb. pressure, followed by double filtraion. This filtrate is 'soil extract'.

**** To make coconut water a fresh coconut was used to collect its water. The coconut water was boiled and double filtration was done to retain the precipitated protiens, if any. This filtrate is known in as 'Coconut water'.

The seeds of *Rhynchostylis retusa* germinate in 2–3 weeks while that of *Aerides multiflora* takes about 3-4 weeks to germinate. The germinated

Table 2

Additives	Concentration	Seed germination	Protocorm formation
Basal medium*		70–80%	Normal
Peptone	1.0 gm/l	enhanced	enhanced
Kinetin	0.5–2.5 mg/l	enhanced	enhanced
I.A.A.	0.5–2.5 mg/l	enhanced	enhanced
2, 4–D	1.0–5.0 mg/l	inhibited	inhibited
Urea	0.50 gm/l	enhanced	inhibited
,,	0.75 gm/l	enhanced	inhibited
,,	1.00 gm/l	enhanced	inhibited

* Mitra *et al.* medium.

seeds initially were whitish to pale in colour, develops the chlorophyll and within a month differentiated in spherical protocorms with basal absorbing hairs.

In both the taxa the protocorms soon develop juvenile shoots, however, the root formation was delayed for considerable time.

DISCUSSION

Orchid-seeds are smallest than that of any other group of angiosperms. They possess undifferentiated embryos and lack endosperm. The germination is defined as, the embryo enlarges, testa ruptures and embryo emerges out. This stage is called spherule and is followed by the formation of protocorm. The subsequent development results formation of basal absorbing hairs, shoots and roots.

In the present investigation, Mitra *et al.* medium was found better than anyother medium. The seed germination was 70-80% for *Aerides multiflora* and *Rhynchostylis retusa*. But when basal medium was supplemented with different combinations of auxin and other growth regulators, show suppression or acceleration in seed germination and protocorm growth, as under.

Faster growth of protocorm and higher seed germination than the normal was found in presence of peptone when added 1.0 gm/l. This was observed in both the taxa.

Kinetin at low concentration gives higher percentage of seed germination and batter protocorm growth.

I.A.A. at low concentration also stimulated maximum seeds to germinate and subsequently better protocorm growth.

2,4-D was always found to be inhibitory to seed germination and protocorm growth in *Aerides multiflora* and *Rhynchostylis retusa*. According to Sharma and Tandon [27], both germination and seedling development

were generally inhibited by 2,4–D. In *coelogyne punctulata,* Vij *et at.* [33] observed that 2,4–D was also inhibitory for inflorescence segment culture of *Saccolabium calceolare.* While many orchids do not need auxin for seed germination, 2, 4–D. either inhibits germination or stimulates callusing of seeds [18]. However in *Pinus gerardiana* (gymnosperm) Mehra and Anand [16] found 2,4-D, when supplemented with MS (Murashige & Skoog) medium and 15% coconut water,produced solid and yellowish callus with good growth but 1 ppm kinetin was required to turn callus greenish.

Urea was found to be inhibitory to protocorm growth, delayed chlorophyll formation and reduced leaf in new seedlings, but favour the germination of seeds which is 80-90%.

Several modifications have been made by many scientist to the culture media, adding tomato juice, banana pulp, orange juice, which give different response to germination and seedling growth [4].

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STUDIES ON GROWTH AND SPORULATION OF *PHYTOPHTHORA NICOTIANAE* var. *PARASITICA* (DASTUR) WATERHOUSE FROM DIFFERENT HOSTS IN RELATION TO THEIR MATING TYPES

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ABSTRACT

The present study was conducted to find out the relationship between cultural characteristics, i.e. growth and sporulation with their compatibility types of various isolates of *Phytophthora nicotianae* var. *parasitica* (Dastur) Waterhouse, isolated from *Psidium guajava* and *Piper betle* hosts. Results obtated during this investigation revealed that three isolates of this pathogen, i.e. KKR1 (1976), YNR (1979) and UCN (1980) isolated from guava host were of A^1 type while other guava isolates, i.e. KKR2 (1979), KKR3 (1980) and Poona (359) were of A^2 type and showed a prominent

correlation with the cultural characteristics, i e. size of colony, aerial mycelium, No. of sporangia and their shape.

Isolates of this pathogen isolated from *Piper betle* host, i.e. from *Piperina* of (Saugar, M.P.) were of A^1 type while from *Piperina* II (Jabalpur, M.P. and *Piperina* (Bindki, Fatehpur.) were of A^2 types but no correlation could be found between cultural characteristics and compatibility types of these three isolates, isolated from *Piper betle*. However, the number of sporangia, showed their trend according to their compatibility types.

Key words : *Phytophthora nicotianae* var. *parasitica*, Guava, Mating types, Compatibility, Sporangia.

INTRODUCTION

Guava *(Psidium guajava)* is the fourth most important crop of India in area and production. It has important place in human diet. Guava fruit crop is attacked by several fungal pathogens and *Phytophthora nicotiana* var. *parasitica* (Dastur) Waterhouse which is a severe pcthogen and attack on guava fruit during rainy season in northern part of India. Incidence of the disease has been reported by various workers (Mitra [1]; Singh *et al*. [2]; Singh *et al*. [3]).

A survey was conducted in some guava growing areas from 1976 to 1980 and various isolates were collected from guava fruits. These isolates were identified as *Phytophthora nicotianae* var. *parasitica* (Dastur) Waterhouse. Besides this, *Phytophthora nicotianae* var. *parasitica* (Dastur) Waterhouse also attacks on various other important crops like *Piper betle*, orange and tomato, ete. The species of *Phytophthora parasitica* var. *piperina* responsible for leaf and foot rot of *Piper betle* is now known as *P. nicotianae* var. *parasitica*. *Piper betle* is an important crop and is being used by a large majority of people in India, Pakistan, Burma, and Sri Lanka, etc. It is well–known fact that there are two mating types of this

species existing (Prasad [4]), Since morphological characteristics of sporangia serve the basis for taxonomy in *Phytophthora nicotianae* var. *parasitica* (Dastur) Waterhouse.

Present investigation was undertaken to examine the morphological variation in various isolates of *P. nicotianae* var. *parasitica* (Dastur) Waterhouse, isolated from different hosts i.e. *Psidium guajava* and *Piper betle* host, if any and relationship of morphology with their compatibility types one host and if any relationship with other host.

MATERIALS AND METHODS

Both A^1 and A^2 compatibility types of *Phytophthora nicotianae* var. *parasitica* (Dastur) Waterhouse have been obtained from guava fruits of Kurukshetra and Yamuna Nagar and also from *Piper betle* of Saugar (M.P.) and Bindki Fatehpur (U.P.) Their compatibility type was established with the help of standard A^1 and A^2 vide No. 591 and 732 respectively, supplied by Prof. G.A. Zentmyer, University of California, Riverside, USA. These isolates were selected for present studies because they produce abundant sporangia. Culture of these isolates was maintained on PDA and CAM medium.

Equally plated (CA) plates were centrally inoculated with the discs of 8 mm in diameter which were taken from the edges of the six-day old colony of the fungus. Inoculated petridishes were incubated at 26±1°C in BOD incubator for growth of colony. Observations were made for radial growth after six days of incubation.

Discs of 8 mm diameter were taken from edges of colony and were floated in sterilized water (18 ml) and were incubated in BOD at 26±1°C. Water was regularly changed at an interval of 6–8 hours. Sporangia were counted under 100X, while size ot sporangia was measured at 400X magnification. Mycelial mats were taken from floating discs and washed with dionised water and mounted with lectophenol and cotton blue for hyphae measurement. Experiments were carried out in the presence of flourescence light.

RESULTS

Results obtained during the present study on growth and sporangial formation of different isolates of *Phytophthora nicotianae* var. *parasitica* isolated from *Psidium guajave* and *Piper betelvine* are presented in Table 1. Maximum average diameter of colony was recorded in Yamuna Nagar isolate of 1979, followed by KKR1 (1976), UNI (1980), KKR2 (1979), and Poona isolate (359) and *Piperina* (II) of Jabalpur. An abundant aerial mycelium was recorded in Yamuna Nagar isolate (1979) followed by KKR1 (1976) and UN1 (1980). In Poona isolate and KKR2 (1979), aerial mycelium was of moderate status. Poor mycelial growth was found in the isolate of *Piperina* Saugar (M.P.).

Maximum average number of sporangia were produced in YNR (1979) followed by KKR1 (1976), KKR2 (1976) and UN1 (1980). Poor sporangial formation was found in Poona isolate. Sporangial numbers of isolate of Piperina host were slightly lesser than guava isolate.

Results on sporangial morphology revealed that larger and papillate sporangia were found in YNR (1979), while in KKR1 (1976) some of them were oval type, In KKR2 (1979), the sporangia were mostly spherical but there were some papillate sporangia also and ratio in between these two was 68–70% to 30–35% respectively. Sporangia of both the shapes were present in Poona isolate. Results have been presented in Table.

DISCUSSION

During investigating morphological studies of different isolates, it has been found that diameter of colony and aerial mycelium were nearly the same in Kurukshetera isolate 1 (1976), UNC1 (1980) and Yamuna Nagar isolate 2 (1979), while it was smaller in both Poona and Kurukshetra isolate 2 (1979). The differences in growth pattern of the colony in two groups can be explained due to diffrent compatibility types as Kurukshetra isolate 1 (1976), Yamuna Nagar isolate (1979) and UNI (1930) are of A^1 type while Kurukshetra isolate 2 (1979) and Poona isolate are of A^2 type. The

relationship of the colony size with mating types has not been found in case of isolate, isolated from *Piper betelvine* host. This may be due to climatic factors affecting the nature of host and pathogen or both. As it is much obvious that all the three isolates of *Piper betle* vine are quite far from each and other.

Relationship with number of sporangia and compatibility types of *Phytophthora nicotianae* var. *parasitica* in guava isolate has been observed. In isolate of A^1 type, number of sporangia were more than A^2 type and this relation also has been found in isolates of *Piper betelvine, Phytophthora nicotianae* var *parasitica.*

Morphological studies of sporangia revealed that sporangia were papillate in KKR1 (1975), YNR isolate (1979), UNC1 (1980), but in the other three isolates of guava showed non-papillate sporangia. The shape of the sporangia may be correlated with the compatbility types of the pathogen. KKR1 (1976), YNR (1979) and UNC1 (1980) are having of same type of sporangia as they are of A^1 type, while KKR2 (1979) and Poona isolate (359) are having similar non-papillate sporangia as they are of A^2 types.

Papillated sporangia were frequently present in the isolate of *P. nicotianae* var *parasitica* on isolate of *Piper betel* from Saugar. In the isolates obtained from *Pipe: betel* (Bindki) and Piperina II (Jabalpur , generally papillae were absent. The pattern has been observed in the same fashion as found in case of Guava isolates.

Summarily in this study, a clear relationship in between growth pattern and compatibility types has been found in the guava isolates of *Phytophthora nicotianae* var. *parasitica.*

ACKNOWLEDGEMENT

The author is higly grateful to Prof. R. S. Mehrotra for his noble guidance during this investigation. Thanks are also recorded to Prof.

G. A. Zentmyer for sending the standard cultures of A^1 and A^2 types of *Phvtophthora parasitica* and also Poona isoltes. Further, Iextend my thanks to CSIR for financial assistance in the form of a Senior Research Fellowship and Post-Doctoral Researed Fellowship during the course of this investigation.

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Table 1

Growth and Morphological Characteristics of Colonies and Compatibility types of various Isolates of *P. nicotianae* var. *parasltica*.

Isolates	Average diameter of colony after six days of incubation (in cm.)	Gradation for aerial mycelium	Average No. of sporangia	Comatibility
KKR1 (1976)	9.5	++++	121	A^1
KKR2 (1979)	9.0	++++	89	A^2
KKR3 (1980)	8.4	+++	97	A^2
UCN1 (1980)	9.4	++++	100	A^1
YNR (1979)	9.5	++++	137	A^1
359 (Poona)	9.0	++++	54	A^2
Piperina II	9.0	++++	80	A^2
Piperina Saugar	8.4	++	90	A^1
Piperina from Bindki	8.5	+++	80	A^2

KKR1 (1976) isolatcd in 1976 from Kurukshetra.
KKR2 (1979) isolated in 1979 from Gurukul, Kurukshetra.
KKR3 (1980) isolated in 1980 from Gurukul, Kurukshetra.
UCN1 (1980) isolated in 1980 from Univeresity.
YNR (1979) isolated in 1979 from Yamuna Nagar (Ambala).
Poona (359) isolated (Borrowed from stock collection of California).

Piperina II (Jabalpur).
Piperin (Saugar).
Piperina (Bindki).

Gradation for Mycelium
++++ (Abundant)
+++ (Moderate)
++ (Poor)

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol 2 (1988), 25–34

A SEARCH FOR SOLAR – WEATHER RELATIONSHIP THROUGH LIGHTNIG ACTIVITY CAUSING ATMOSPHERIC IONIZATION

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ABSTRACT

The short term fluctuations in solar proton flux and lightning index are correlated. A power spectral analysis shows similar periods of fluctuation in the two parameters. On the basis of these results a mechanism of corre-

lation between thunderstorm activity and solar activity is given and thus the previous observations of long and short term of correlations are given a possible explanation

Key wcrds and phases : Solar proton flux, Magnetic Solar sector boundary, Lightning index, Power spectrum and Ionization.
Physics and stronomy Classification Scheme (PACS 1977) : 94.30,94.40k

INTRODUCTION

The effect of sun on atmospheric electricity has been studied long before by Watson [13]. Markson [6] reported a positive correlation between the thunderstorm occurance frequency and solar activity for high latitudes and a negative correlation in equatorial latitudes. The strongest positive correlation was found in Siberia. The correlation coefficients between thunder storms and sunspots for different geographic locations were found by Brooks [2]. The 5 year running mean of both annual mean sunspot numbers and the annual index of lightning were in phase for the years 1930 to 1970 [2]. Markson [7] explained this correlation in terms of the ionizing radiation from the sun which increased the atmospheric resistivity thereby causing decrease in the atmospheric electric field. The change in atmospheric clectric field influences the charge generation process [8] which in turn gives high lightning activity.

The present study provides an additional support to the mechsniams proposed by Markson [7] and Herman and Goldberg [4,5].

ANALYSIS OF THE DATA

The present paper examines the short term fluctuations in solar wind proton flux at the outer boundary of the magnetosphere and in the lightning activity observed at the high latitude ground station (Bonn, 50.44 N 9.04 E). For the former we utilize the published data [10] and for the later the data obtained at stockert near Bonn. The lightning activity was obtained by the old version of the VLF Atmospherics Analyser which has now been installed

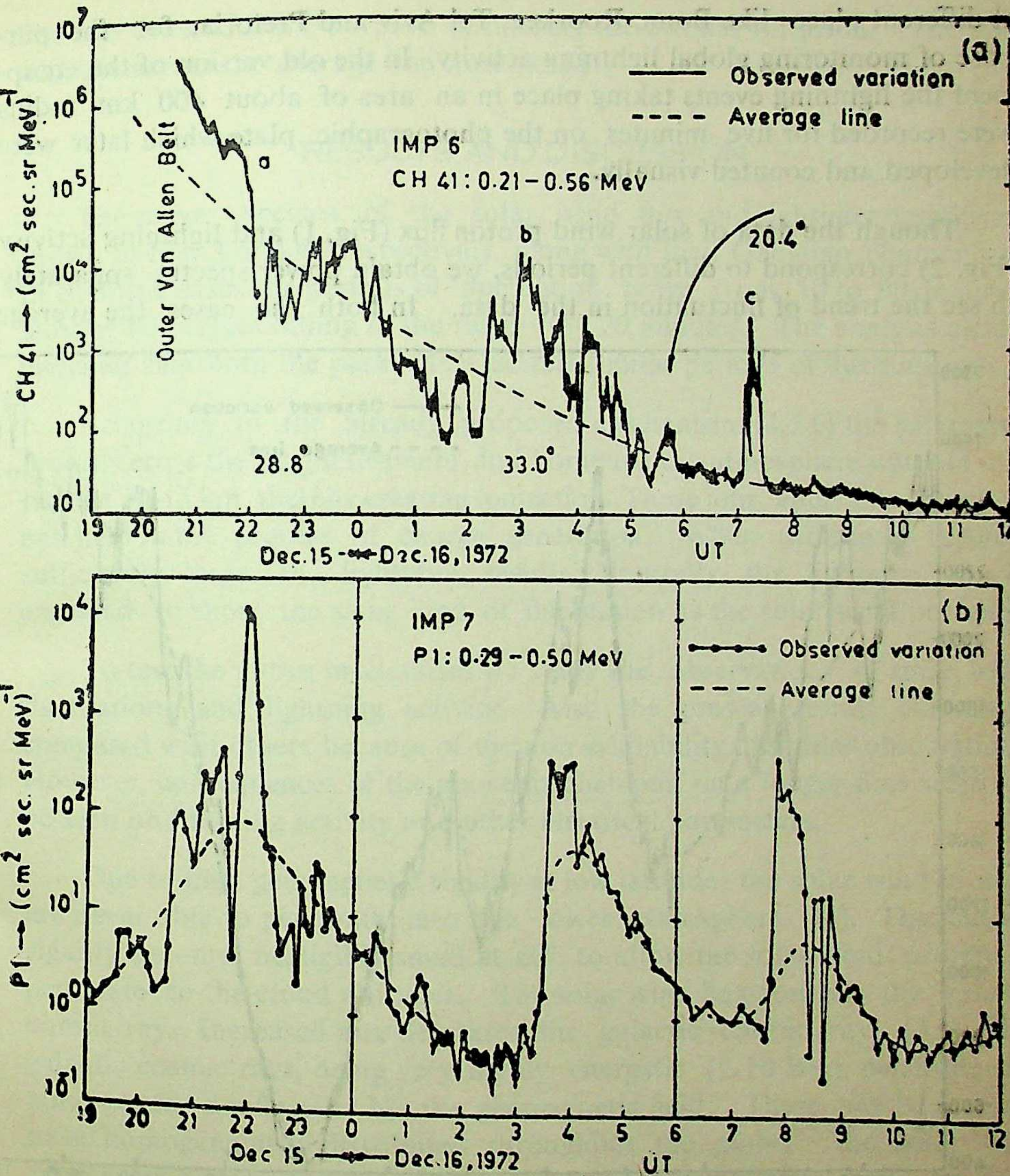


Fig 1 Observations of proton flux on Dec. 15-16, 1972 [10] with average line drawn, by
(a) Imp 6 outside the dusk trapping boundary.
(b) Imp 7 in the distant dusk magneto tail

at different places like Bonn, Roorkee, Tel Aviv and Pretoria, for the purpose of monitoring global lightning activity. In the old version of the equipment the lightning events taking place in an area of about 400 km radius were recorded for five minutes on the photographic plate which later were developed and counted visually.

Though the data of solar wind proton flux (Fig. 1) and lightning activity (Fig. 2) correspond to different periods, we obtain power spectra spparately to see the trend of fluctuation in the data. In both the cases the average

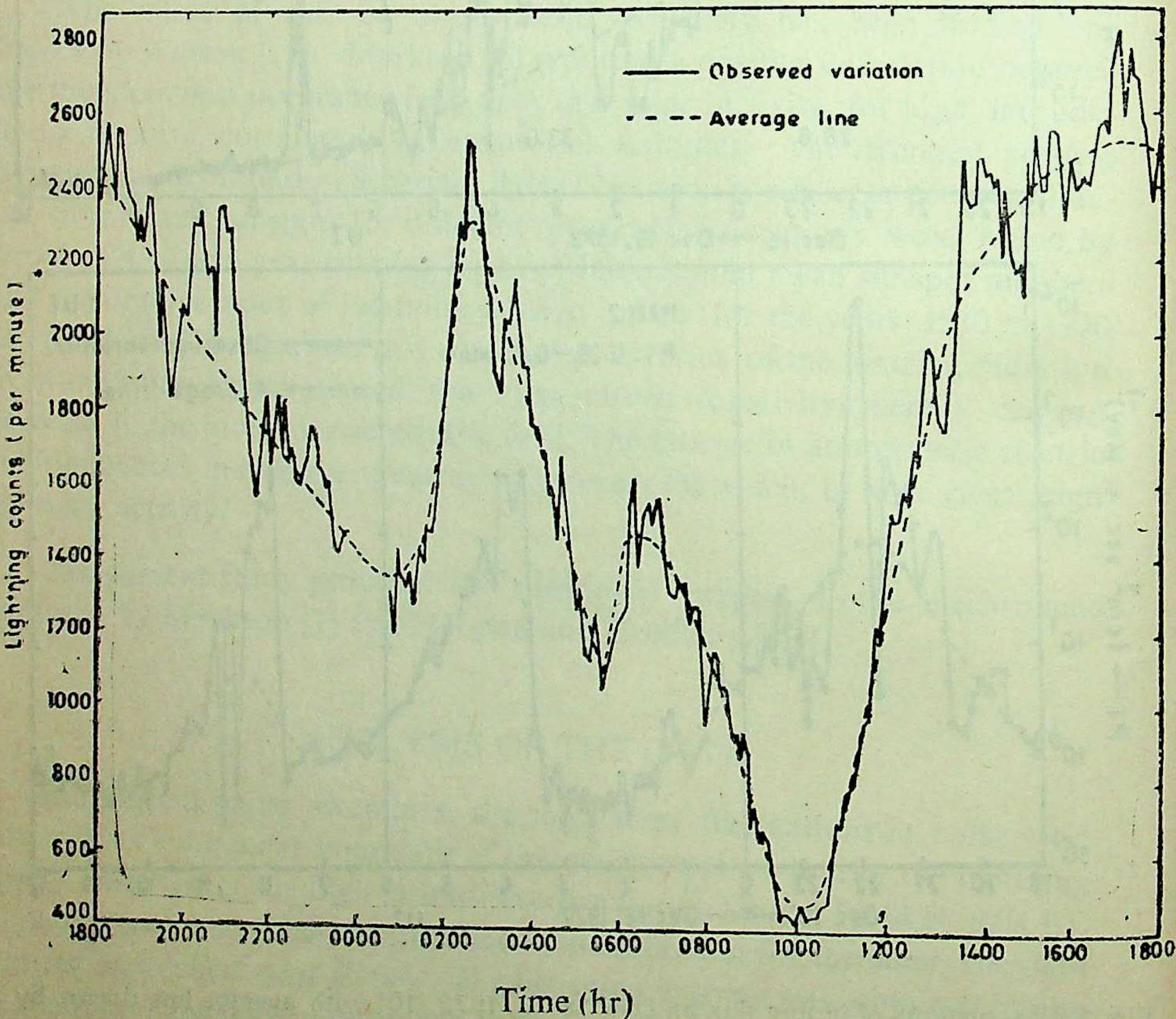


Fig. 2 Observations of lightning activity at Stockert near Bonn, F.R.G.,

line was drawn for the purpose of filtering the diurnal variation. The fluctuations across the average line were noted.

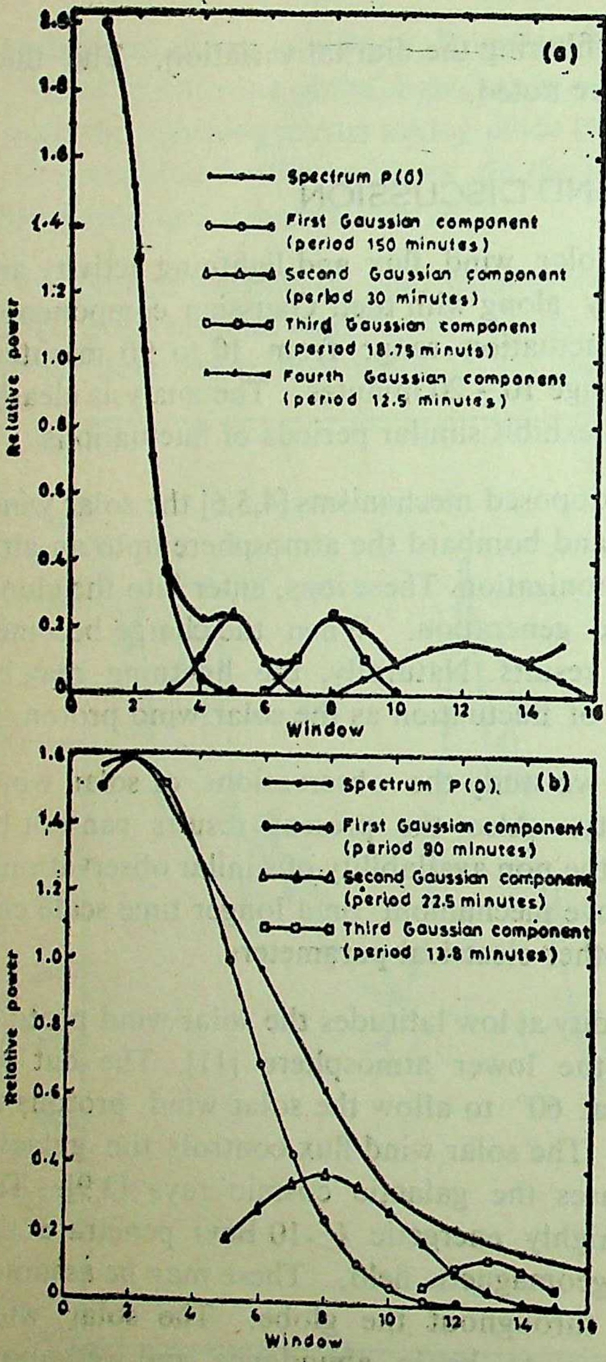
RESULTS AND DISCUSSION

The power spectra of the solar wind flux and lightning activity are shown in Fig. 3 and 4, respectivaly along with their Gaussian components. In both the cases the periods of fluctuation range from 10 to 90 minutes, most of the values falling in the range 10 – 20 minutes. The analysis clearly indicates that both the parameters exhibit similar periods of fluctuations.

According to the already proposed mechanisms [4,5,6] the solar wind protons cross the magnetosphere and bombard the atmosphere upto an altitude of 10-15 km thereby creating ionization. These ions, enter into the cloud and help in the process of charge generation. When the charge becomes sufficiently large, the lightning results. Naturally, the lightning may be expected to show the same kind of fluctuation as the solar wind proton.

To test the above mechanism we study the observations of solar wind fluctuations and lightning activity. Also the present results can not be compared with others because of the non availability of similar observations. However, consequences of the above mechaniom on a longer time scale can be seen on lithtning activity and other electrical parameters.

Due to high geomagnetic rigidity at low latitudes the solar wind protons are never able to penetrate into the lower atmosphere [11]. The cut off rigidity becomes negligibly small at 60° to allow the solar wind protons to penetrate to the cloud altitudes. The solar wind flux controls the galactic comsic rays. Increased flux decreases the galactic cosmic rays [3,9]. The galactic cosmic rays, being very highly energetic (>10 Bev) penetrate the atmosphere unaffected by the geomagnetic field. These may be assumed to be homogeneously distributed throughout the globe. The solar wind protons at high latitudes are known to be in abundance and well above the galactic cosmic ray level. Thus, one can say that the contribution to the atmospheric ionization at low latitudes is due to galactic cosmic rays and that at high latitudes it is due to the solar wind protons.



If the mechanism of Herman and Goldberg [4,5] is really operative, the solar activity (producing increased amount of solar protons) and the lightning activity and atmospheric electric field (maintained by the lightning) at low and high latitudes must be anticorrelated. It should be in phase with solar activity at high latitudes while at low latitudes it should be out of phase. In fact Bauer [1] found that the potential gradi ent in the British Isles, France and Spain varies in phase with sunspot numbers.

The authors are monitoring lightning activity at Rookee using a new version of the Atmospherics Analyser. The further analysis of the data for this low latitude will be given later.

CONCLUSION

The power spectral analysis of lightning activity and the solar wind proton flux observed at the outer boundary of the

Fig. 3 Power Spectrum of short term fluctuations in proton flux, across the average line drawn in, (a) Fig. 1 a and (b) Fig. 1 b.

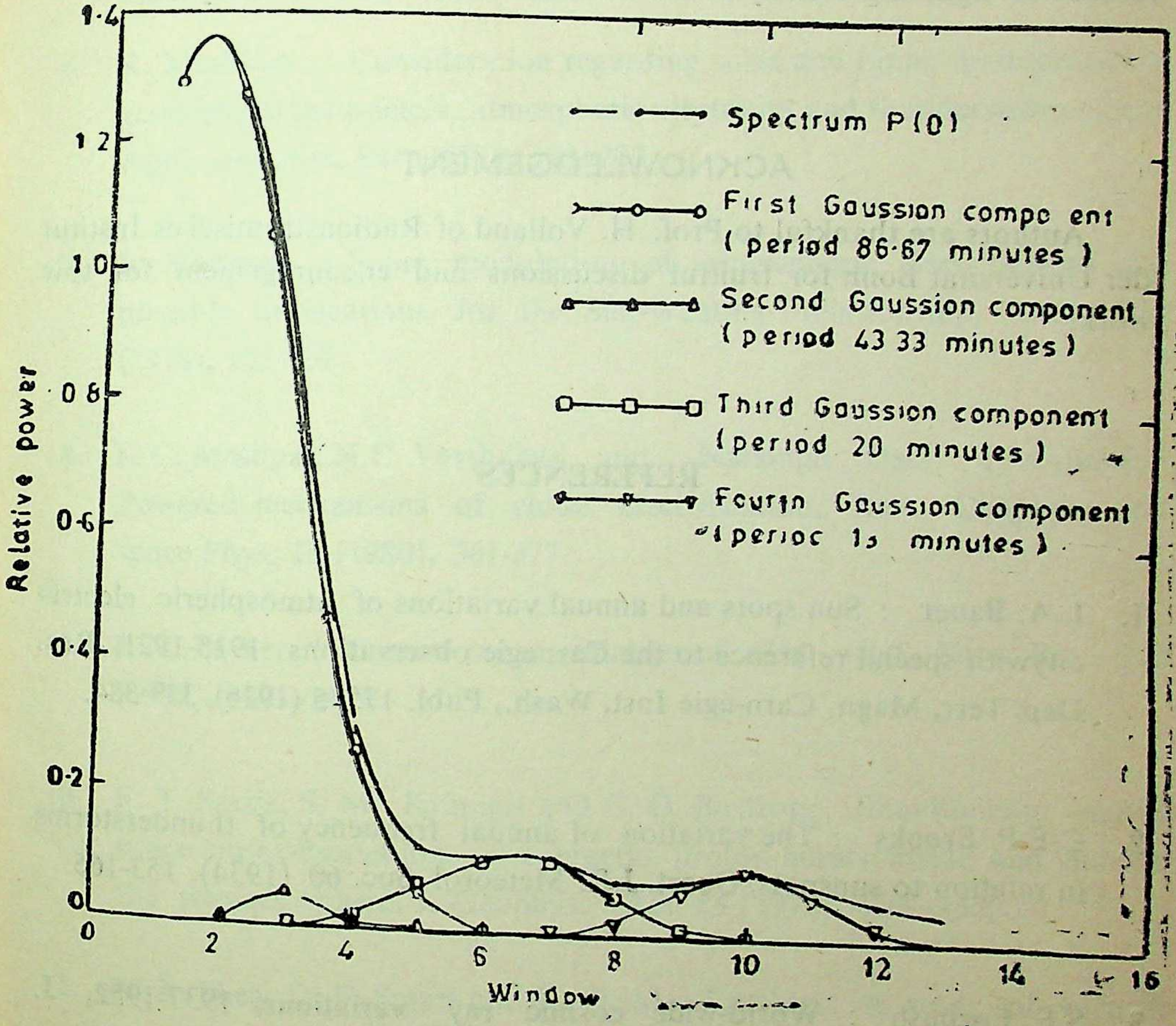


Fig. 4 Power spectrum of short term fluctuations of lightning index across the average line drawn in Fig. 2.

magnetosphere show approximately the same periods of fluctuation. The study provides an additional support to the mechanism proposed by Markson [7] and Herman and Goldberg [4,5] However, for concrete study the records of lightning activity at low latitudes should also be analysed.

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COINCIDENCES AND FIXED POINTS OF MULTIVALUED MAPPINGS IN PROBABILISTIC METRIC SPACES

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ABSTRACT

Let A denote an arbitrary (nonempty) set and $C(X)$ the set of closed subsets of a Menger space $(X, \mathfrak{F}, t)$. The main result of this paper is a coincidence theorem for the mappings $P\ Q : A \rightarrow C(X)$ and $T : A \rightarrow X$. This result extends the fixed point theorems for multivalued contraction mappings in metric spaces to Menger spaces. The conditions for the existence of a unique common fixed point of a pair of multivalued contraction mappings on a Menger space have also been investigated.

Key words & Phrases : Multivalued contraction mapping, Probabilistic metric space, Menger space, Coincidence, Fixed point.

AMS (MOS) Mathematics Subject Classification (1980) : 54H25, 54C60

§ 0 INTRODUCTION

The concept of probabilistic metric spaces (*PM*–spaces) was introduced by K. Menger [9], and was, later, investigated by himself [10] and others. However, due to the pioneering work of Schweizer and Sklar [20], varied attention of mathematicians was drawn, in the sixties, towards the *PM*-spaees. Sehgal [21] initiated the study of contraction mappings in *PM*-spaces and fixed point theorems for contractions, and local contractions (resp. generalized contractions) were proved by Sehgal and Bharucha-Reid [22] (resp. Ćirić [2]). Egbert [4] introduced the concept of probabilistic diameter of a set, and extended the definition of Hausdorff metric to probabilistic spaces.

The theory of multivalued mappings has wide applications to game theory, mathematical economics, optimal control theory, multifunctional equations etc. A substantial place, in the theory of multivalued mappings, mainly due to its applications to functional equations, is claimed by the study of fixed points of multivalued contraction mappings (see for instance. [23]).

Hadźić [6] and Pai and Veeramani [13] seem to initiate the study of fixed points of multivalued mappings in *PM*- spaces. However, none of the results [6,13] is an exact extension of multivalued contraction principle in metric space, first proved by Nadler [11] and subsequently generalized by others, (see, for instance, Ćirić [1], Iséki [7], Kaulgud and Pai [8] and Reich [16]).

The § 1 of this paper is devoted to notations and definitions to be used in the sequel. Using Egbert's definition of probabilistic Hausdorff metric (cf. Definition 3) we, in § 2, introduce the notion of multivalued contraction mappings (abbreviated as *mvcm*) and *generalized mvcm* in *PM*-spaces and obtain fixed point theorems for such mappings. In fact, the main result (Theorem 1) of § 2, is a coincidence theorem for two multivalued mappings and a single-valued mapping in *PM*-spaces. This result, apart from including several known coincidence and fixed point theorems for single-valued

and multivalued mappings, extends fixed point theorems for *mvcm* in metric spaces to *PM*-spaces. Since unique fixed point theorems play an important role in many applications, the intent of § 3 is to investigate conditions under which a pair of multivalued mappings in *PM*-spaces may have a unique common fixed point.

§ 1 PRELIMINARIES

Let (M,d) be a metric space and D the Hausdorff metric induced by d. Further, let $P: M \rightarrow CB(M)$, the set of nonempty bounded subsets of M. Then P is a multivalued contraction mapping on M if

$$D(Pu, Pv) \leq \alpha\, d(u, v)$$

for every u, v in M and some nonnegative number $\alpha < 1$. (Here it may be mentioned that $CB(M)$ was later on replaced by $C(M)$, the closed subsets of M ; see, for instance, Covitz and Nadler [3] and Ćirić [1]).

We shall follow the following notations and definitions to be used in the sequel, Let $(X, \mathfrak{F}, t)$ be a Menger space. Then for any nonempty subsets A, B of X, we define distribution functions G and L as follows :

$$G_{A,B}(\ ) = \underset{k<x}{\text{lub}} \{ \underset{\substack{u \in A \\ v \in B}}{\text{glb}} F_{u,v}(k) \} \tag{1}$$

$$L_{A,B}(x) = \underset{k<x}{\text{lub}} \{ \underset{\substack{u \in A \\ v \in B}}{\text{lub}} F_{u,v}(k) \}. \tag{2}$$

We shall prefer to write $L_{u,B}(x)$ in place of $L_{A,B}(x)$, when $A = \{u\}$. Note that $L_{u,B}(x)$ denotes the probability that the ordinary distace between the point u and the set B is less than X. Also, for $v \in B$,

$$F_{u,v}(x) \leq L_{A,B}(x), \; x > 0.$$

The function $E_{A,B}$, defined by

$$(3) \qquad E_{A,B}(X) = \underset{k<x}{\text{lub}}\ t\{\underset{u\in A}{\text{glb}}\ [\underset{v\in B}{\text{lub}}\ F_{u,v}(k)],\ \underset{v\in B}{\text{glb}}\ [\underset{u\in A}{\text{lub}}\ F_{u,v}(k)]\},$$

is called the probabilistic distance between A and B, (Egbert [4, Def. 5]), and is a distribution function [4]. For any subsets A, B of X,

$$G_{A,B}(X) \leq E_{A,B}(X) \leq L_{A,B}(X).$$

With 't=min' and A, B singletons, clearly $E_{A,B} = L_{A,B} = G_{A,B}$.

Recall that 't=min' is the strongest possible universal t [20].

Throughout this paper $C(X)$ stands for the set of closed subsets of X. If E is defined for $A, B \in C(X)$ then Egbert has shown that $(C(X), \mathscr{E}, t)$ is a Menger space [4, Th. 18]. The value of $\mathscr{E}$ at $A, B \in C(X)$, as usual, will be denoted by $E_{A,B}$. In analogy with the Hausdorff metric space $(C(X), \mathscr{E}, t)$ may be called Egbert-Hausdorff Menger space or simply EHM-space induced by $(X, \mathfrak{F}, t.)$

We shall need a number of lemmas. The first two are obvious from the above definitions.

LEMMA 1. Let A, B be in $C(X)$. Then for all u in A and for some h, b in $(0, 1)$, there exists a v in B such that

$$F_{u,v}(h^{-b}X) \geqslant E_{A,B}(X) \qquad \text{for } X > 0.$$

LEMMA 2. Let A be in $C(X)$ and b in $(0, 1)$. Then for every u in A, there exists a v in A such that for $X > 0$.

$$G_{u,A}\,(b^{-1}x) \geqslant F_{u,v}(x) \text{ and } E_{u,A}\,(b^{-1}x) \geqslant F_{u,v}(x).$$

LEMMA 3. Let A, B and C be nonempty subsets of X. Then for a fixed u in A, $w \in B$ and $v \in C$,

$$L_{u,B}\,(x+y) \geqslant t\,\{F_{u,v}(x),\; L_{C,B}(y)\}, \qquad \text{for } x>0,\ y>0.$$

PROOF. Let x be an arbitrarily fixed positive number, and u a point in X. Then, since for any $v \in C \subset X$, $w \in B \subset X$ and $s > 0$,

$$F_{u,w}\,(x+s) \geqslant t\,\{F_{u,v}(X),\; F_{v,w}(s)\},$$

we have

$$\operatorname*{lub}_{w\in B} F_{u,w}\,(x+s) \geqslant t\,\{\operatorname*{lub}_{v\in C} F_{u,v}(x),\; \operatorname*{lub}_{\substack{v\in C\\ w\in B}} F_{v,w}(s)\}.$$

So

$$L_{u,B}\,(x+y) = \operatorname*{lub}_{X+s<X+y} \{\operatorname*{lub}_{w\in B} F_{u,w}(x+s)\}.$$

$$= \operatorname*{lub}_{s<y} \{\operatorname*{lub}_{w\in B} F_{u,w}(X+s)\}$$

$$\geqslant \operatorname*{lub}_{s<y} \{t\,[\operatorname*{lub}_{v\in C} F_{u,v}(x),\; \operatorname*{lub}_{\substack{v\in C\\ w\in B}} F_{v\,w}(s)]\}$$

$$\geqslant t(F_{u,v}(x), L_{C,B}(y)).$$

LEMMA 4. Let A, B and C be nonempty subsets of X. Then for a fixed u in A, w in B and v in C,

$$G_{u,B}\,(x+y) \geqslant t\{F_{u,v}(x),\; G_{v,B}(y)\}, \qquad \text{for } x>0,\ y>0.$$

PROOF. Let X be an arbitrary fixed positive number, and u a point in X. Then, since for any $v \in C \subset X$, $w \in B \subset X$ and $s > 0$,

$$F_{u,w}(x+s) \geqslant t\{F_{u,v}(x) : F_{v,w}(s)\},$$

we have

$$\operatorname*{glb}_{w \in B} F_{u,w}(x+s) \geqslant t\{F_{u,v}(x), \operatorname*{glb}_{w \in B} F_{v,w}(s)\}.$$

So

$$\begin{aligned} G_{u,B}(x+y) &= \operatorname*{lub}_{x+s<x+y} \{\operatorname*{glb}_{w \in B} F_{u,w}(x+s)] \\ &= \operatorname*{lub}_{s<y} \{\operatorname*{glb}_{w \in B} F_{u,w}(x+s)\} \\ &\geqslant \operatorname*{lub}_{s<y} t\{F_{u,v}(x), \operatorname*{glb}_{w \in B} F_{v,w}(s)\} \\ &= t\{F_{u,v}(x), \operatorname*{lub}_{s<y} [\operatorname*{glb}_{w \in B} F_{v,w}(s)]\} \\ &= t\{F_{u,v}(x). G_{v,B}(y)\}. \end{aligned}$$

§ 2 MAIN RESULTS

Let $(X, \mathfrak{F}, t)$ be a Menger apace, where t is continuous and satisfies $t(x, x) \geqslant x$ for every $x \in [0, 1]$ and E the Egbert Hausdorff metric induced by F. Let P: $X \to C(X)$. Consider the following conditions :

For every u, v in X and some $h \in (0, 1)$,

(4) $$E_{Pu,\,Pv}\,(hx) \geqslant F_{u,\,v}\,(x)\ ;$$

(5) $$E_{Pu,\,Pv}\,(hx) \geqslant \min\{F_{u,\,v}\,(x),\ L_{u,\,Pu}(x),\ L_{v,\,Pv}(x)\ L_{u,\,Pv}(2x)\ L_{v,\,Pu}(2X)\}.$$

Conditions (4) and (5) will be called multivalued contraction and generalized multivalued contraction respectively, on X, and P is called *mvcm* and *generalized mvcm* respectively. Obviously, (5) includes (4.)

THEOREM 1. Let $(x, \mathfrak{F}, t)$ be a Menger space, where t is continuous and satisfies $t(x,x) \geqslant x$ for every $x \in [0, 1]$, and let P and Q be mappings from an arbitrary (nonempty) set A to $C(X)$. If there exists a mapping $T{:}A \to X$ and a positive number $h < 1$ such that $T(A)$ is a complete subspace of X with $P(A) \cup Q(A) \subset T(A)$, and

(6) $$E_{Pu,\,Qv}\,(hx) \geqslant \min\{\ F_{Tu,\,Tv}\,(x),\ L_{Tu,\,Pu}\,(x),\ L_{Tv,\,Qv}\,(x),\ L_{Tu,\,Qv}\,(2x),\ L_{Tv,\,Pu}\,(2x)\ \}$$

for all u,v in A ; then P, Q and T have a coincidence, i.e. there exists a point z in A such that $Tz \in (Pz \cap Qz)$.

PROOF. Let u_0 be an arbitrary point in A, and b a fixed number in $(0, 1)$. We construct two sequences $\{\ u_n\ \}$ of points of A and $\{\ v_n\ \}$ of points of $T(A)$ in the following manner :

Since $P(A) \cup Q(A) \subset T(A)$, we can choose $v_1 = Tu_1 \in Pu_0$. If $Pu_0 = Qu_1$, choose $v_2 = Tu_2 \in Qu_1$ such that $v_2 = v_1$; and if $Pu_0 \neq Qu_1$, choose $v_2 = Tu_2 \in Qu_1$ such that

$$F_{v_1, v_2}(h^{-b} x) \geqslant E_{Pu_0, Qu_1}(x).$$

Further, if $Qu_1 = Pu_2$, choose $v_3 = Tu_3 \in Pu_2$ such that $v_3 = v_2$; and if $Qu_1 \neq Pu_2$, choose $v_3 = Tu_3 \in Pu_2$ such that

$$F_{v_2, v_3}(h^{-b} x) \geqslant E_{Qu_1, Pu_2}(x).$$

In general we choose,

$v_{2n+2} = Tu_{2n+2} \in Qu_{2n+1}$ such that

$v_{2n+2} = v_{2n+1}$ if $Pu_{2n} = Qu_{2n+1}$, and

$F_{v_{2n+1}, v_{2n+2}}(h^{-b} x) \geqslant E_{Pu_{2n}, Qu_{2n+1}}(\ )$ if $Pu_{2n} \neq Qu_{2n+1}$;

and

$v_{2n+1} = Tu_{2n+1} \in Pu_{2n}$ such that

$v_{2n+1} = v_{2n}$ if $Pu_{2n} = Qu_{2n-1}$, and

$F_{v_{2n}, v_{2n+1}}(h^{-b} x) \geqslant E_{Pu_{2n}, Qu_{2n-1}}(x)$ if $Pu_{2n} \neq Qu_{2n-1}$.

Now, either $F_{v_{2n}, v_{2n+1}}(h^{-b} x) = 1$ if $Pu_{2n} = Qu_{2n-1}$

or

$F_{v_{2n}, v_{2n+1}}(h^{-b} x) \geqslant E_{Pu_{2n}, Qu_{2n-1}}(x)$, otherwise.

So, when $Pu_{2n} \neq Qu_{2n-1}$, by (6),

$$F_{v_{2n}, v_{2n+1}}(h^{1-b} x) \geqslant E_{Pu_{2n}, Qu_{2n-1}}(hx)$$

$$\geqslant \min \{ F_{v_{2n}, v_{2n-1}}(x), L_{v_{2n}, Pu_{2n}}(x),$$

$$L_{v_{2n-1}, Qu_{2n-1}}(x), L_{v_{2n}, Qu_{2n-1}}(2x),$$

$$L_{v_{2n-1}, Pu_{2n}}(2x) \}$$

$$\geqslant \min \{ F_{v_{2n}, v_{2n-1}}(x), F_{v_{2n}, v_{2n+1}}(x),$$

$$F_{v_{2n-1}, v_{2n+1}}(2x) \}.$$

Since $F_{v_{2n-1}, v_{2n+1}}(2x) \geqslant \min \{ F_{v_{2n-1}, v_{2n}}(x), F_{v_{2n}, v_{2n+1}}(x) \}$

we have

$$F_{v_{2n}, v_{2n+1}}(h^{1-b} x) \geqslant F_{v_{2n-1}, v_{2n}}(x), \text{ when } Pu_{2n} = Qu_{2n-1}.$$

Thus, in either of the cases $Pu_{2n} \neq Qu_{2n-1}$ and $Pu_{2n} = Qu_{2n-1}$,

we obtain

(7) $$F_{v_{2n+1}, v_{2n+2}}(h^{1-b} x) \geqslant F_{v_{2n}, v_{2n+1}}(x).$$

Similarly, in either of the cases $Pu_{2n} = Qu_{2n+1}$ and $Pu_{2n} \neq Qu_{2n+1}$,

(8) $$F_{v_{2n+1}, v_{2n+2}}(h^{1-b} x) \geqslant F_{v_{2n}, v_{2n+1}}(x).$$

In view of (7) and (8), we have

$$F_{v_n, v_{n+1}}(h' x) \geqslant F_{v_{n-1}, v_n}(x), \quad n=1, 2\ldots\ldots,$$

where $h' = h^{1-b}$.

Since $h' < 1$, $\{ v_n \}$ is a Cauchy sequence in $T(A)$, and hence converges to some p in $T(A)$. So there exists a point z in X such that

$$Tz = p.$$

Let $u_{Tz}(\varepsilon, \lambda)$ be a neighbourhood of Tz. Since $v_n \to Tz$, for $\varepsilon > O$, $\lambda > O$ there exists an $N = N(\varepsilon, \lambda)$ such shat

(9) $$F_{Tz, v_{2n+2}}\left(\frac{1-b}{2}\varepsilon\right) > 1-\lambda \quad \text{for all n} \geqslant N.$$

Now by Lemma 3.

(10) $$L_{Tz, Px}(\varepsilon) \geqslant \min \left\{ F_{Tz, v_{2n+2}}\left(\frac{1-h}{2}\varepsilon\right), L_{Qu_{2n+1}, Pz}\left(\frac{1+h}{2}\varepsilon\right)\right.$$

By (6),

$$L_{Pz, Qu_{2+1}}\left(\frac{1+h}{2}\varepsilon\right) \geqslant E_{Pz, Qu_{2n+1}}\left(\frac{1+h}{2}\varepsilon\right)$$

$$\geqslant \min \left\{ F_{Tz, Tu_{2n+1}}\left(\frac{1+h}{2h}\varepsilon\right), L_{Tz, Pz}\left(\frac{1+h}{2h}\varepsilon\right),\right.$$

$$L_{Tu_{2n+1},\, Qu_{2n+1}}\left(\frac{1+h}{2h}\varepsilon\right),\ L_{Tz,\, Qu_{2n+1}}\left(\frac{1+h}{h}\varepsilon\right),$$

$$L_{Tu_{2n+1},\, Pz}\left(\frac{1+h}{h}\varepsilon\right)\}$$

$$\geqslant \min\{F_{Tz,\, v_{2n+1}}\left(\frac{1+h}{2h}\varepsilon\right),\ L_{Tz,\, Pz}\left(\frac{1+h}{2h}\varepsilon\right),$$

$$F_{v_{2n+1},\, v_{2n+2}}\left(\frac{1+h}{2h}\varepsilon\right),\ F_{Tz,\, v_{2n+2}}\left(\frac{1+h}{h}\varepsilon\right)$$

$$F_{v_{2n+1},\, Tz}\left(\frac{1+h}{2h}\varepsilon\right),\ L_{Tz,\, Pz}\left(\frac{1+h}{2h}\varepsilon\right)\}$$

Therefore from (9) and (10),

$$L_{Tz\,.\,Pz}(\varepsilon) > 1-\lambda \text{ for } n \geqslant N.$$

Hence, $Tz \in Pz$. Similarly $Tz \in Qz$,

Thus $Tz \in (Pz \cap Qz)$.

COROLLARY 1. Let $(X, \mathfrak{F}, t)$ be a Menger space, where t is continuous and satisfies $t(x,x) \geqslant x$ for every $x \in [0, 1]$, and $P, Q, T : A \rightarrow X$ such that for all u,v in A and $h \in (0, 1)$,

$$F_{Pu,\, Qv}(hx) \geqslant \min\{F_{Tu,\, Tv}(x),\ F_{Pu,\, Tu}(x),\ F_{Qv,Tv}(x),$$

$$F_{Pu,\, Tv}(2x),\ F_{Qv,\, Tu}(2x)\};$$

$T(A)$ is a complete subspace of X; and $P(A) \cup Q(A) \subseteq T(A)$ Then P,Q and T have a coincidence.

PROOF. Theorem 1 with $P, Q : A \rightarrow X$, immediately reduces to this Corollary.

Setting $P=Q$, $A=X$ and $T=I$ in Theorem 1, we have the following result :

COROLLARY 2. Let $(X, \mathfrak{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t(x,x) \geqslant x$ for every $x \in [0, 1]$. If P is a generalized mvcm on x, then P has a fixed point in X.

PROPOSITION. Let (M,d) be a complete metric space and D the Hausdorff metric induced by d. If $P : X \rightarrow C(X)$ such that

$$(11) \quad D(Pu,Pv) \leqslant h.\max\{d(u,v), d(u,Pu), d(v,Pv), \tfrac{1}{2} d(u, Pv), \tfrac{1}{2} d(v, Pu)\}$$

for every u, v in X and some $h \in (0, 1)$; then P has a fixed point in X.

The above result includes several fixed point theorems for multivalued contractions in metric spaces proved in [11], [14], [15], [16], [19] and elsewhere. Note that Corollary 2 is an exact extension of this proposition.

COROLLAY 3. Let $(X, \mathfrak{F}, t)$ be a Menger space where t is continuous and satisfies $t(x,x) \geqslant x$ for every $x \in [0, 1]$ and let P, Q be multivalued mappings from an arbitrary (nonempty) set A to $C(X)$. If there exists a mapping $T : A \rightarrow X$ and a positive number $h < 1$ such that $T(A)$ is a complete subspace of X with $P(A) \cup Q(A) \subseteq T(A)$, and

$$E_{Pu, Qu}(hx) \geqslant F_{Tu, Tu}(x)$$

for all u, v in A ; then P, Q and T have a coincidence.

This result is an exact extension of a recent coincidence theorem of [12, Th. 2] to PM-spaces, while [12, Th. 2] extends and unifies Goebel's coincidence theorem [5, Th. 1] and the Banach contraction principle for mvcm in metric spaces (cf. [11, Th. 5] or [3, Cor. 1]).

§ 3

THEOREM 2. Let $(X, \mathfrak{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t(x,x) \geqslant x$ for every $x \in [0, 1]$, and $P, Q : X \to C(X)$ such that

$$(12) \qquad G_{Pu, Qv}(hx) \geqslant \min\{F_{u,v}(x), G_{u,Pv}(x), G_{v,Qv}(x), G_{u,Qv}(2x), G_{v,Pu}(2x)\}$$

for every u, v in X and some constant $h < 1$. Then P and Q have a unique common fixed point, i.e., there exists a unique $z \in X$, such that $z \in (Pz \cap Qu)$.

PROOF. Let b be a fixed number in $(0, 1)$. Then in view of Lemma 2, there exist mappings $f, g : X \to X$ such that $fu \in Pu$ $gu \in Qu$, for all $u \in X$, and

$$G_{u, Pu}(h^{-b} x) \geqslant F_{u, fu}(x), \quad G_{u, Qu}(h^{-b} x) \geqslant F_{u, gu}(x) \quad \text{for all } u \in X.$$

By (12),

$$F_{fu, gv}(h^{1-b} x) \geqslant G_{Pu, Qv}(h^{1-b} x)$$

$$\geqslant \min\{F_{u,v}(h^{-b} x), G_{u,Pu}(h^{-b} x), G_{v,Qv}(h^{-b} x)$$

$$G_{u,Qv}(2h^{-b} x), G_{v,Pu}(2h^{-b} x)\}$$

$$\geqslant \min\{F_{u,v}(h^{-b} x), G_{u,Pu}(h^{-b} x),$$

$$G_{v,Qv}(h^{-b} x), F_{u,v}(h^{-b} x), G_{v,Qv}(h^{-b} x)$$

$$F_{v,u}(h^{-b} x), G_{u,Pu}(h^{-b} x)\}$$

$$\geqslant \min \{ F_{u,v}(x) \;\; F_{u,fu}(x,) \;\; F_{v,gv}(x) \}.$$

Then, in view of Corollary 1, with $A = X$ and $T = I$, f and g have a common fixed point, which is also a common fixed point of P and Q.

To prove that P and Q have a unique common fixed point, let there be two distinct points z and w such that $z \in (Pz \cap Qz)$ and $w \in (Pw \cap Qw)$. First we note, by the definition of f, that.

$$G_{z,Pz}(h^{-b}x) \geqslant F_{z,fz}(x) = 1.$$

Similarly

$$G_{w,Pw}(h^{-b}x) = G_{z,Qz}(h^{-b}x) = G_{w,Qw}(h^{-b}x) = 1.$$

So

$$F_{z,w}(x) \geqslant G_{Pz,Qw}(x)$$

$$\geqslant \min \{ F_{z,w}(x/h), G_{z,Pz}(x/h), G_{w,Qw}(x/h),$$

$$G_{z,Qw}(2x/h) \;\; G_{w,Pz}(2x/h) \}$$

$$\geqslant \min \{ F_{z,w}(x/h), 1, 1, F_{z,w}(x/h),$$

$$G_{w,Qw}(x/h), \; F_{w,z}(x/h), G_{z,Pz}(x/h) \}$$

$$= F_{z,w}(x/h),$$

yielding $z = w$.

A similar proof follows for the next result.

THEOREM 3. Let $(X, \mathfrak{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t(x,x) \geqslant x$ for every $x \in [0, 1]$. Further, let $P, Q : (X) \to C(X)$ such that

$$G_{Pu,\,Qv}(hx) \geqslant \min\{F_{u,v}(x),\ E_{u,\,Pu}(x),\ E_{v,\,Qv}(x),\ E_{u,\,Qv}(2x),$$
$$E_{v,\,Pu}(2x)\}$$

for all u, v in X and some constant $h \in (0,1)$. Then P and Q have a unique common fixed point.

THEOREM 4. Let $(X, \mathfrak{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t(x,x) \geqslant x$ for every $x \in [0,1]$. Further, let P, Q: $X \to C(X)$, such that

$$G_{Pu,\,Qv}(hx) \geqslant \min\{F_{u,v}(x),\ G_{u,\,Pu}(x),\ G_{v,\,Qv}(x),\ L_{u,\,Qv}(2x),\ L_{v,\,Pu}(2x)\} \tag{13}$$

for all u, v in X and some constant $h \in (0,1)$. Then P and Q have a common fixed point in X, i. e., there exists a z in X such that $z \in (Pz \cap Qz)$.

PROOF. Since $L_{u,\,Qv}(2x) \geqslant G_{u,\,Qv}(2x)$ and $L_{v,\,Pu}(2x) \geqslant G_{v,\,Pu}(2x)$, condition (13) implies (12). Hence the proof follows from Theorem 2.

Theorems 2-4 unify and extend to PM-spaces, a number of fixed point theorems for multivalued mappings in metric spaces (see, for instance, [1], [7], [11], [17], [18] and [19]).

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APPROXIMATING FIXED POINTS OF MULTIVALUED MAPS

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ABSTRACT

Recently Kuhfittig has studied the Mann iteration process for certain classes of multivalued operators, in fact point-compact operators. It is known that if an operator *T* satisfies a certain contractive condition and the sequence of Ishikawa iterates converges then it converges to a fixed point of *T*. In this paper an Ishikawa type iteration scheme for point-closed (multivalued) maps is defined and the result is extended to this new scheme, thus extending the results of Rhoades and Naimpally *et al.*

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Key words and Phrases : Mann iteration, Ishikawa iteration, multivalued operator, fixed point.

§ 1

Recently Kuhfittig [5] has extended Krasnosel'skii's iteration procedure [4] to Mann iteration process (cf. [2] and [12]) for multivalued maps. and more recently Naimpally and Singh [7] have shown that, for maps satisfying very general contractive conditions, if the sequence of Shiro Ishikawa iterates [3] converges, it converges to a fixed point of the map (see also [9]).

In this paper we define an Ishikawa type iteration scheme for a multivalued map, and show that the sequence converges to a fixed point of the map satisfying a general type of contractive condition, if the sequence under the defined scheme converges at all.

Consistent with [6, p. 620], for A, B in $CL(X)$, the nonempty closed subsets of a normed linear space X, and $\varepsilon > 0$, let us follow the following notations :

$$N(\varepsilon, A) = \{ x \in X : \| x - a \| < \varepsilon \text{ for some } a \in A \},$$
$$E_{A,B} = \{ \varepsilon > 0 : A \subset N(\varepsilon, B),\ B \subset N(\varepsilon, A) \},$$

$$H(A, B) = \begin{cases} \inf E_{A,B}, \text{ if } & E_{A,B} \neq \varnothing \\ +\infty \quad, \text{ if } & E_{A,B} = \varnothing ; \end{cases}$$

and, for $x \in X$,

$$D(x, A) = \inf_{a \in A} \| x - a \| .$$

The following lemma is well-known (see, for instance, Rus [11]).

Lemma : Let $q > 1$ and $A, B \in CL(X)$. Then for every $x \in A$, there exists $y \in B$ such that $d(x, y) \leq q\, H(A, B)$.

We shall discuss the following iteration scheme for a map T from a closed convex subset C of X to $CL(C)$.

(1.1) $x_o \in C$;

(1.2) $y_n = b_n p_n + (1 - b_n) x_n$, $n \geqslant 0$, $p_n \in Tx_n$;

(1.3) $x_{n+1} = (1 - a_n) x_n + a_n q_n$, $n \geqslant 0$,

wherein, in view of the above lemma, $q_n \in Ty_n$ is such that

(1.4) $\| p_n - q_n \| \leqslant k^{-\lambda} H(Tx_n , Ty_n)$

for a given k in $(0, 1)$ and some λ in $(0, 1)$;

(1.5) $0 \leqslant a_n , b_n \leqslant 1$ for all n ;

(1.6) $\underline{\lim}\ a_n > 0$.

We remark that if T is a map from C to C then the demand (1.4) is trivially satisfied, since $k^{-\lambda} > 1$.

Let $O(x_n , T , y_n) = \{ x_i , y_i : i = 0, 1, 2, \dots \}$.

We shall call $O(x_n , T , y_n)$ the *orbit* of T for the scheme defined by (1.1)–(1.6). Evidently, for $T : C \to C$, $O(x_n , T, y_n)$ defines the Ishikawa iteration scheme (see, for instance, [7], [9]).

§ 2 RESULTS

Theorem 2.1 : Let C be a closed convex subset of a normed linear space X, and $T : C \to CL(C)$. Further, let the sequence $\{x_n\}$ defined previously converge to a point z. If T satisfies

(2.1) $$H(Tx, Ty) \leqslant k \max \{ \| x - y \| , D(x, Tx), D(y, Ty), D(x, Ty) + D(y, Tx) \}$$

for every $x, y \in O(x_n , T , y_n) \cup \{z\}$, then $z \in Tz$.

Proof : Since $x_n \to z$ and $\{a_n\}$ is bounded away from zero, it follows from $\| x_{n+1} - x_n \| = a_n \| x_n - q_n \|$ and $\| q_n - z \| \leqslant \| z - x_n \| + \| x_n - q_n \|$ that $\| x_n - q_n \|$ and $\| q_n - z \|$ both tend to zero. Moreover, form (1.2) and the triangle inequality,

$$\| x_n - y_n \| = b_n \| x_n - p_n \| \leqslant \| x_n - p_n \|,$$

$$\| x_n - p_n \| \leqslant \| x_n - q_n \| + \| q_n - p_n \|,$$

$$\| y_n - q_n \| \leqslant \| y_n - x_n \| + \| x_n - q_n \|$$

$$\leqslant 2 \| x_n - q_n \| + \| q_n - p_n \|,$$

and

$$\| y_n - p_n \| = (1 - b_n) \| x_n - p_n \| \leqslant \| x_n - p_n \|.$$

Hence from

$$\| p_n - q_n \| \leqslant k^{-\lambda} H(Tx_n, Ty_n)$$

$$\leqslant k^{-\lambda} \max \{ \| x_n - y_n \|, D(x_n, Tx_n), D(y_n, Ty_n), D(x_n, Ty_n) + D(y_n, Tx_n) \}$$

$$\leqslant k' \max \{ \| x_n - p_n \|, \| y_n - q_n \|, \| x_n - q_n \| + \| y_n - p_n \| \}, \; k^{1-\lambda} = k' \text{ (say)},$$

it follows that

$$\| p_n - q_n \| \leqslant 2k' (1-k')^{-1} \| x_n - q_n \|.$$

Thus $\| p_n - q_n \| \to 0$, $\| x_n - p_n \| \to 0$ and $\| p_n - z \| \to 0$ as $n \to \infty$.

From (2.1),

$$\begin{aligned} D(z, Tz) &\leqslant \| z - p_n \| + D(p_n, Tz) \\ &\leqslant \| z - p_n \| + H(Tx_n, Tz) \\ &\leqslant \| z - p_n \| + k \max \{ \| x_n - z \|, D(x_n, Tx_n), \\ &\qquad D(z, Tz), D(x_n, Tz) + D(z, Tx_n) \} \\ &\leqslant \| z - p_n \| + k \max \{ \| x_n - z \|, \| x_n - p_n \|, \\ &\qquad D(z, Tz), \| x_n - z \| + D(z, Tz) + \| z - p_n \| \}, \end{aligned}$$

which yields, in the limit, $z \in Tz$.

Remark 2. 1 : If T is a (single-valued) map from C to C, then Theorem 2.1 presents a generalization of a result of Rhoades [9, Theorem 9]. Theorem 2.1 with $T : C \to C$ is also a variant of a result of Naimpally and Singh [7, Theorem 1.2] , since, in their result, the single-valued analogue of (2.1) is required to be satisfied for all x, y in C.

It is well-known that $T : C \to CL(C)$ satisfying

(2.2d) $$H(Tx, Ty) \leqslant k \max \{ \| x - y \|, D(x, Tx), D(y, Ty), \tfrac{1}{2} (D(x, Ty) + D(y, Tx)) \}$$

for every x, y in C possesses a fixed point in a complete space C. In fact, this result, in an orbitally complete metric space, was proved by Ćirić [1].

For $T : C \to CL(C)$, consider the following conditions :

(2.2a) $$H(x, Tx) + H(y, Ty) \leqslant a \| x - y \|, 1 \leqslant a < 2 ;$$

(2.2b) $$H(x, Tx) + H(y, Ty) \leqslant b \{D(x, Ty) + D(y, Tx) + \| x - y \| \}, \quad 1/2 \leqslant b < 2/3 ;$$

(2.2c) $$H(x, Tx) + H(y, Ty) + H(Tx, Ty) \leqslant c \{ D(x, Ty) + D(y, Tx) \}, \quad 1 \leqslant c < 3/2.$$

Conditions (2.2a)-(2.2d) for a single-valued map T (on a metric space) were first studied by Pal and Maiti [8]. Using the Pal and Maiti's conditions, Rhoades [10, Theorem 2] showed that if the sequence of Mann iterates (for Pal and Maiti's map on C) converges, it converges to a fixed point of the map. Naimpally and Singh [7] extended Rhoades' result to the Ishikawa iteration scheme $O(x_n, T, y_n)$ for $T : C \to C$. Our next theorem extends their result [Theorem 1. 4, 7] to the scheme defined previously (cf. (1. 1)-(1.6)).

Theorem 2.2 : Let C be a closed convex subset of a normed linear space X, and $T : C \to CL(C)$. Further, let the sequence $\{x_n\}$ defined previously converge to a point z. If, for each $x, y \in O(x_n, T, y_n) \cup \{z\}$, T satisfies at least one of the conditions (2.2a) - (2.2d), then $z \in Tz$, provided

(2.2e) $\quad k^{-\lambda} \leqslant 2$ or $\underline{\lim}\, b_n > (c - 1 - k^{\lambda}) / (c + 1)$.

Proof : We shall make frequent use of the fact that, for any $x \in C$ and any $y \in B$, $\| x-y \| \leqslant H(x, B)$, $B \in CL(C)$, [1, p. 269].

Since $p_n \in Tx_n$ and $q_n \in Ty_n$,

$$2 \| p_n - q_n \| \leqslant (\| x_n - p_n \| + \| y_n - q_n \|) + \| x_n - q_n \| + \| y_n - p_n \|$$

$$\leqslant H(x_n, Tx_n) + H(y_n, Ty_n) + \| x_n - q_n \| + \| y_n - p_n \| .$$

Moreover, as in the proof of Theorem 2.1,

$$\| x_n - q_n \| \to 0 \text{ and } \| q_n - z \| \to 0.$$

If, for $x = x_n$ and $y = y_n$, (2.2a) is satisfied, then :

$$2\,\|p_n - q_n\| \leqslant a\,\|x_n - y_n\| + \|x_n - q_n\| + \|y_n - p_n\|$$

$$\leqslant a\,b_n\,(\|x_n - q_n\| + \|q_n - p_n\|) + \|x_n - q_n\|$$

$$+ (1-b_n)(\|x_n - q_n\| + \|q_n - p_n\|),$$

i. e. $\|p_n - q_n\| \leqslant t_1\,\|x_n - q_n\|$, where

$$t_1 = (2 + (a-1)\,b_n)\,/\,(1 - (a-1)\,b_n).$$

If, for $x = x_n$ and $y = y_n$, (2.2b) holds, then :

$$2\,\|p_n - q_n\| \leqslant b\,(D(x_n, Ty_n) + D(y_n, Tx_n) + \|x_n - y_n\|)$$

$$+ (\|x_n - q_n\| + \|y_n - p_n\|)$$

$$\leqslant (b+1)(\|x_n - q_n\| + \|y_n - p_n\|) + b\,\|x_n - y_n\|,$$

giving

$$\|p_n - q_n\| \leqslant t_2\,\|x_n - q_n\|, \text{ where}$$

$$t_2 = (2 + 2b - b_n)\,/\,(1 - b + b_n).$$

If, for $x = x_n$ and $y = y_n$, (2.2c) holds, then :

since $\|p_n - q_n\| \leqslant t\,H(Tx_n, Ty_n)$ with $t = k^{-\lambda}$,

$$(2t+1)\,\|p_n - q_n\| \leqslant ct\,(D(x_n, Ty_n) + D(y_n, Tx_n))$$

$$+ t(\|x_n - q_n\| + \|y_n - p_n\|)$$

$$\leqslant t(c+1)(\|x_n - q_n\| + \|y_n - p_n\|),$$

giving

$$\| p_n - q_n \| \leqslant t_3 \| x_n - q_n \|, \quad \text{where}$$

$$t_3 = t(c+1)(2 - b_n) / (1 + t(1 - c(1 - b_n) + b_n)).$$

Finally if, for $x = x_n$ and $y = y_n$, (2.2d) holds, then :

$$\| p_n - q_n \| \leqslant tk \max \{ \| x_n - y_n \|, D(x_n, Tx_n), D(y_n, Ty_n),$$

$$\tfrac{1}{2}(D(x_n, Ty_n) + D(y_n, Tx_n)) \}$$

$$\leqslant tk(2 \| x_n - q_n \| + \| q_n - p_n \|),$$

that is

$$\| p_n - q_n \| \leqslant t_4 \| x_n - q_n \|, \text{ where } t_4 = 2tk/(1 - tk).$$

Hence, for $x = x_n$ and $y = y_n$, we have

$$(2.3) \qquad \| p_n - q_n \| \leqslant \max \{ t_1, t_2, t_3, t_4 \} \| x_n - q_n \|.$$

Clearly $t_4 > 0$, and t_1, t_2 are positive for any b_n in $[0, 1]$.

We examine t_3. For any b_n in $[0, 1]$, t_3 is positive if

$$1 + t(1 - c(1 - b_n) + b_n) > 0$$

i. e. if $\quad 1 + t(1 - 3(1 - b_n)/2 + b_n) > 0,$

(since c may be closer to $3/2$),

i. e. if $\quad 1 + t(1 - 3/2) > 0$, i. e. if $t < 2$.

Evidently, $t_3 > 0$ for $t = 2$ and for any b_n in $[0, 1]$.

If $t > 2$, then t_3 is positive, if

$$1 + t(1 - c(1 - b_n) + b_n) > 0$$

i. e. if $b_n > ((c - 1)\, t - 1) / (tc + t) = (c - 1 - 1/t) / (c + 1)$.

So, in view of (2.2e), t_3 is positive. Thus, from (2.3),

$\| p_n - q_n \| \to 0$, $\| x_n - p \| \to 0$ and $\| p_n - z \| \to 0$ as $n \to \infty$.

Now we show that z is a fixed point of T. We examine the cases (2.2a)–(2.2d) for $x = x_n$ and $y = z$. In case (2.2a),

$$\| x_n - p_n \| + H(z, Tz) \leqslant H(x_n, Tx_n) + H(z, Tz)$$

$$\leqslant a \| x_n - z \| . \tag{2.4a}$$

Similarly, in case (2.2b),

$$\begin{aligned}\| x_n - p_n \| + H(z, Tz) &\leqslant b(D(x_n, Tz) + D(z, Tx_n) + \| x_n - z \|) \\ &\leqslant b(\| x_n - z \| + D(z, Tz) \\ &\quad + \| z - p_n \| + \| x_n - z \|) \\ &\leqslant b(2 \| x_n - z \| + \| z - p_n \| + H(z, Tz))\end{aligned}$$

that is

$$(1 - b)\, H(z, Tz) \leqslant 2b \| x_n - z \| - (1 - b) \| z - p_n \| . \tag{2.4b}$$

In case (2.2c),

$$\begin{aligned}2H(z, Tz) &\leqslant H(z, Tz) + \| z - x_n \| + H(x_n, Tx_n) + H(Tx_n, Tz) \\ &\leqslant \| z - x_n \| + c(D(x_n, Tz) + D(z, Tx_n)) \\ &\leqslant \| z - x_n \| + c(\| x_n - z \| + D(z, Tz) + \| z - p_n \|) \\ &\leqslant (c + 1) \| z - x_n \| + c\, H(z, Tz) + c \| z - p_n \|\end{aligned}$$

that is

$$(2.4c) \quad (2-c)\, H(z, Tz) \leqslant (c+1)\ \| z - x_n \| + c\, \| z - p_n \| .$$

Finally, in case of (2.2d) ,

$$D(z, Tz) \leqslant \| z - p_n \| + H(Tx_n , Tz)$$

$$\leqslant \| z - p_n \| + k \max \{ \| x_n - z \| , D(x_n, Tx_n), D(z, Tz), \tfrac{1}{2}(D(x_n , Tz) + D(z, Tx_n)) \}$$

$$(2.4d) \qquad \leqslant \| z - p_n \| + k \max \{ \| x_n - z \| , \| x_n - p_n \| , D(z, Tz), \tfrac{1}{2} (\| x_n - z \| + D(z, Tz) + \| z - p_n \|) \}.$$

So, in the limit, we get from each of (2.4a) — (2.4c),

$H(z, Tz) = 0$, and from (2. 4d), $D(z, Tz) = 0$. Hence $z \in Tz$.

Remark 2. 3 : Following Ćirić [1] and Pal and Maiti [8], it can be shown that T mapping from a closed convex subset C of a Banach space to $CL(C)$ and satisfying, for each x, y in C, at least one of the conditions (2. 2a) — (2.2d) possesses a fixed point.

Remark 2.4 : Since, while defining the iteration scheme (1.1)-(1.4), for a given k one can choose λ in $(0,1)$ such that $k^{-\lambda} \leqslant 2$, the condition (2.2e) may be dropped from Theorem 2.2.

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फार्म—४

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका, खण्ड २, अंक १-२

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राष्ट्रीयता	:	भारतीय
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ब्रह्मचर्येण तपसा देवा मृत्युमपाघ्नत

स्थापना १९५७ विक्रमाब्द

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EFFECT OF TEMPERATURE ON DEVELOPMENT OF CYCAS OVULE ROT AT HARDWAR

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ABSTRACT

Present paper deals with the effect of temperature on incidence of Cycas ovule rot at Hardwar. Pathogenecity experiment of Cycas ovule rot due to *Fusarium moniliforme* and *F. moniliforme* var. *intermedium* was conducted at 30°C and 25°C temperature respectively. Larger sized and dark brown coloured lesions were produced by both pathogens at 30°C but lesions were of smaller size and of light brown colour at 25°C. Maximum growth of both pathogens was found at 30°C which was 430 mg (DW), 8.6 cm (diametre of colony) & 570 mg (DW), 7.6 cm. respectively.

Key Words : Cycas, Ovule Symptoms, growth, Pathogen.

INTRODUCTION

Cycas a well known ornamental plant, includes twenty species and grows from Japan to Australia and Madagoskar (Africa). It is a slow growing long lived and evergreen palm like plant. Six species of *Cycas* are found in our country and *C. revoluta* Thenub is the commonest species. Generally female plants of this species are found only in northern India.

The incidence of the disease was observed in the month of October 1986 on the ovules of *C. revoluta* grown in the Botanical garden of

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Gurukul Kangri University Hardwar. Persistence of infection on ovule has been noted upto month of March in year 1987, 88 and 89 regularly. About sixty to seventy percent ovules were infected with the disease. Dry ovule rot of Cycas was a new disease in this country and the pathogen involved in the disease has not been reported from country till now (1,2,4,5). The pathogens were identified as *Fusarium moniliforme* and *Fusarium moniliforme* var. *intermedium* by CMI, Ferryland, Kew surrey, England.

As temperature and humidity are very important factors in development of any disease and growth of the pathogen therefore present investigation was undertaken to find out the relationship of temperature with incidence of disease and growth of the pathogen.

MATERIALS AND METHODS

Medium sized ovules showing typical characteristics of the disease (dark brown lesion on ovule surface), were collected from the plants and were brought to the laboratory for isolation of the pathogen and further studies. Ovules were surface sterilized by treating with 0.01% $HgCl_2$ solution for 15 seconds to kill the surface microflora. Treated ovules were repeatedly washed 5 to 6 times with sterilized water. These ovules were cut into small pieces with the help of sterilized blade and were inoculated on PDA plates. Certain ovule pieces were placed on PDA with their inner surface. Inoculated petridishes were incubated in BOD incubater at 26±1°C for growth of the pathogen. The pathogens were purified and stocked on PDA slants. Cultures were sent to CMI and were identified as *Fusarium moniliforme* and *Fusarium moniliforme* var. *intermedium*.

These cultures were used for pathogenecity experiment invitro at 25°C and 30°C temperature. Growth of the pathogens (radial and dry weight) was also studied at above mentioned temperature to see the effect of temperature on development of disease and growth of the pathogens.

Pathogenecity Experiments in Vitro at Different Temperatures

Medium sized healthy ovules showing typical characteristic orange colour were used in the present investigation. The ovules along with

megasporophylls were surface sterilized with $HgCl_2$ solution of. 01% for twenty seconds and washed repeatedly with sterilized water. Injuries were made on the ovule surface and injured ovules were inoculated with discs of 8mm. of both pathogens. Discs were taken from edges of six days old colony grown on PDA medium. These inoculated ovules were placed in sterilized enamel trays and were covered with glass beljar. Sterilized wet cotton was placed inside the beljar to provide hundred percent humidity for development of the disease symptoms. Inoculated ovules with both pathogens were incubated at 25°C and 30°C temperature respectively in BOD incubator for development of disease symptoms. Observation was made after 10 days of incubation.

Growth of Pathogens at Different Temperature

The pathogens were grown on potato dextrose liquid broth (PDB) and on PDA medium to find out the effect of temperature on growth of the pathogens. Earlier plated Petridishes were inoculated with the discs of 8 mm. diameter of both pathogens and were incubated in BOD incubatore at 25°C and 30°C for their radial and mycelial growth. The flasks of 250 ml. capacity containing 30 ml. sterilized PDB were inoculated with 8 mm. discs of the pathogens and were incubated at 25°C and 30°C temperature. Diameter of the colony and dry weight of the mycelium was recorded after six days of incubation.

Result and discussion

Temperature has shown profound effect on development of disease symptoms (lesions) on Cycas ovule. It has been found that lesions were much clear, dark brown coloured and larger sized at 30°C than at 25°C in both pathogens i.e. *F. moniliforme* and *F. moniliforme* var. *intermedium* respectively (Table-1).

Maximum diameter of colony as well as dry weight of mycelium in both of pathogens occured at 30°C temperature. Record of dry weight of mycelium showed profound effect of temperature on growth of pathogens (table 2).

Higher growth rate of both pathogens at 30°C is directly related with the incidence of the disease. As both pathogens grew well at 30°C temperature it indicates that pathogens could increase the inoculum potential at 30°C which is an important factor for development of any disease. Day temperature of Hardwar during October and November months was found 30°C in the garden and campus. Symptoms of the disease on Cycas ovule was also observed during October 1986. Luxurient growth of both pathogens at 30°C temperature invitro and incidence of the disease on Cycas ovule at this temperature showed positive correlation in between temperature and disease development. Kulkarni (3) also studied the temperature effects on development of cotton wilt caused by *F. oxysporum, F. vasinfectum* (Atk) Snyder and Hansen and found that 31°C temperature was maximum temperature for disease development.

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TABLE-I

Characterestics of 6ycas ovule rot at different Temperature

Organisms	Incubation Period	Temperature 0°C	Average area of Lesion in cm.	Characterestics of ovule rot
Fusarium moniliforme	10	25	1.00 to 1.5	Generally Circular Semicircular and light in colour
	10	30	2.0 to 3.3	Circular and dark brown colour
F. monili-forme-var. -	10	25	1.0 to 3.3	Circular and dark brown colour.
intermedium	10	30	2.0 to 4.00	Circula and dark brown colours.

TABLE II

Influence of Temperature on growth of *F. moniliforme* and *F. moniliforme* var. *intermedium*

Organisms	Temperature (°C)	Incubation days	Average Diameter of colony in cm.	Dry weight of mycelium in mg.
F. moniliforme	25	6	8.0	400.0
	30	6	8.66	430.00
F. moniliforme var. intermedium	25	6	6.6	552.5
	30	6	7.91	570.0

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EFFECT OF INDUSTRIAL EFFLUENT ON SEED GERMINATION OF SOME AGRICULTURAL CROPS

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ABSTRACT

The present investigation was conducted to find out the effect of industrial effluent of two industries viz. Doon distillery and Indian Drug & Pharmaceutical Limited on seed germination of four agricultural crops i.e. *Triticum vulgare*, *Luffa cylindrica*, *Abelmoschus esculentus* and *Phaseolus mungo*. During this study, maximum percentage of seed germination was recorded in wheat seed i.e. 86.6 at 100 per cent concentration of IDPL effluent (E_2). A gradual fall in percentage of seed germination of wheat crop was recorded with dilution of this effluent. Adverse effect of Doon Distillery effluent (E_1) was recorded in seed germination of the all test crops as compared with control.

Key words :– Pollution, Effluent, Germination, percentage.

INTRODUCTION

Water pollution is a major problem of the world. The problem appears quite serious specially for the surface water which receives various kinds of pollutants through various means. It can be stated truly that various larger and smaller stream and other aquatic bodies in the country are in danger. River Ganga is also facing the same problem. It receives various kinds of pollutants in various form i.e. industrial effluent, sewage and other various kinds of garbage throughout the course.

In Rishikesh and Hardwar region, various industries such as IDPL located at Virbhadra, Rishikesh and BHEL may be considered as big

source of industrial effluent. Discharge of the effluent of these units into Ganga resulted drastic change in water quality (Shanker [8]. Doon Distillery located at KUANWALA, District Dehradun discharges large quantity of its effluent into Reh. This effluent has indirect effect on water quality of Ganga through river song.

The effect of various kinds of pollutants on water quality in different aquatic system has been studied in the country by various workers (see for instance, [2] and [8]. Albeit the use of sewage for irrigation in agricultural crops is in the practice, but industrial effluent is still discharged either on the earth surface or in aquatic system. Discharge of industrial effluent directly in the streams, not only deteriorates the water quality but has adverse effect on aquatic flora and fauna. Shanker *et al.* [9].

The present investigation was undertaken to find out possible utilization of industrial effluent for irrigation. A large area of agricultural land is found in surrounding areas of IDPL and Doon Distillery, and after positive findings, effluent may be used for irrigation purposes so that the effluents are put to economic use in terms of providing irrigation water to fertilizer.

MATERIALS AND METHODS

Industrial effluents of Doon Distillery and IDPL were collected during 1985 at an interval of one month and were evaluated according to APHA for their certain physico-chemical characters (see) [1]. The effluents were further used for the treatment of seeds of test crops. Four different kinds of seeds i.e. *Triticum vulgare* var. RR-21-, *Luffa cylindrica* var. Faizabadi, *Abelmoschus esculentus* and *Phaseolus mungo* were used as test organisms.

Seeds of the above test crops were surface sterilized with. 1% $HgCl_2$ solution and were repeatedly washed with sterilized distilled water for 6-7 times. After proper washings, these seeds were treated in three different concentrations of each effluent i.e. 100 per cent (Pure effluent), 50 per cent and 25 per cent. Three replicates were used for each

concentration. Seeds as such (without treatment) were used for control. Treated seeds were kept in moist chamber, prepared in petridishes of 10 cm diameter with blotting sheets and effluent. Ten seeds were kept in each petridish. These petridishes were kept in room temperature for germination of seed. Percentage of seed germination was recorded after 4 days. Investigation was reconducted three times to confirm the results.

RESULTS AND DISCUSSION

Results of the present investigation have been given in Tabels 1 and 2. Average value of physico-chemical characters of the effluent for the period of a year i.e. 1985 has been given in Table 1. Results revealed that effluent of both industries is highly polluted and contains higher level of total solids, BOD and COD etc. beyond permissible limit (Table 1). Dissolved oxygen was very poor in IDPL effluent and was found always nil in Distillery effluent. Similarly, very low level of pH was recorded in the distillery effluent.

Findings on effect of effluent on seed germination showed that IDPL effluent (E_2) has promotive effect as percentage of seed germination in general except *Phaseolus mungo*. (Table 2). Maximum percentage of seed germination among all the test crops has been recorded in *T. vulgare* i.e. 86.6 at 100 per cent concentration (without dilution). A gradual fall in percentage of seed germination was recorded with dilution of the effluent at 50 and 25 per cent (Table 2). These findings indicate that IDPL effluent contains some nutrients and fulfill the requirement of manures to the germinating and growing crop up to certain extent. Furthermore, the results are encouraging in case of *L. cylindrica* and *A. esculentus* where seed germination was recorded slightly better than the control.

Adverse effect of Doon Distillery effluent (E_1) was recorded on seed germination in all the test crops as compared with control. There was no seed germination at 100% (pure effluent) in *T. vulgare* and *L. cylindrica*, while very poor seed germination was recorded in *P. mungo* and *A. esculentus* (Table 2). Enhancement in percentage of seed germination was found in *A. esculentus* with dilution of effluent at 50 and 25 per cent

(Table 2). However, this did not show any encouragement in case of *T. vulgare* (Table 2).

The poor germination with slight increase with dilution of effluent showed the dilution of various toxic substances which are present in dissolved and suspended particles, BOD and COD etc. as well as this accelerate of pH level of effluent (E_1) i.e. 3.3. Higher concentration of total solids i.e. 25180 mg/lt may be a factor for inhibition of seed germination. Similar finding has been found in seed germination of *Hordium vulgare* by Bahadur and Sharma [3]. Similarly, Sahai and Saxena [7] have also reported harmful effect of effluent on seed germination. Absence of oxygen may also be a cause of inhibition of seed germination in distillery effluent (E_1) as reported by Bahadur and Sharma [3] in *H. vulgare*. Adverse effect of this effluent (E_1) may also be due to another factor i.e. very low level of pH 3.3. As seeds in control were shown in tap water and in general pH of water as found in range 6.8-7.0. The inhibition of seed germination may be due to combined factor rather than any single factor.

Summarily, effluent of IDPL factory (E_2) may be recommended for irrigation for *T. vulgare.*, *L. cylindrica* and *A. esculentus* crops specially for wheat but before to recommend it is highly needed that aforesaid crops should be evaluated for the plant health, yield, fodder quality as well as food and vegetable quality. A biochemical study of the plant and its product is also necessary so that if there is any kind of toxic substances are stored in plant or its product can be estimated quantitatively as well as qualitatively. Albeit in priliminary stage, results of IDPL effluent are promising but cannot be recommended for irrigation till the above information is *not* received experimentally.

TABLE–1

Annual average value of Physico-chemical characteristics of Effluents (1985)

Effluent	pH	Total solids (mg/lt)	D.O. (mg/lt)	BOD (mg/lt)	COD (mg/lt)
E_1	3.3	25180	Nil	7252	15700
E_2	6.9	1200	2.85	189.5	24000

E_1 = Distillery effluent

E_2 = IDPL effluent

TABLE–2

Effect of Effluent on Seed germination of some agricultural crops

Effluent	Treat-ment (%)	Test crop & Percentage of Germination			
		T. vulgare	*L. cylindrica*	*A. esculentus*	*P. mungo*
	100	Nil	Nil	20	0.5
E_1	50	Nil	10	55	17.5
	25	10	43	55	17.5
	100	86.6	56.2	65	Nil
E_2	50	70	59.2	67	Nil
	25	70	36	67	--
Control		53	60	68	75.5

E_1 = Distillery effluent

E_2 = IDPL effluent.

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THEORY OF OLFACTION/GUSTATION BASED ON DIPOLE MOMENT OF MOLECULE

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ABSTRACT

Various Investigators have proposed many theories to explain the phenomena of olfaction [1, 19] and gustation [16], but most of them touched the boundary of the problem. The theory presented in this paper could explain all the aspects of the two phenomena.

The theory helps in understanding the smell and taste (Chemical) senses completley just like there is complete understanding of light and sound senses.

1.00 Introduction : Eyes, ears, nose, tongue and skin are the parts through which sensations are felt by human body. Light and sound sensations felt through eyes and ears have been completely explained by the scientists. However smell and taste sensations felt by the nose and tongue respectively could not be explained satisfactorily by various theories proposed till now. The most popular theory of olfaction has been proposed by Amoore et al. (1) which relates odour quality with shape and size of molecule. Shallenberger et al. [16] proposed a theory of gustation co-relating sweet taste with structure of molecules.

The theory presented in this paper explains completely the two phenomena of chemical senses.

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2.00 Receptors

2.01 Olfactory receptors : There are two types of olfactory receptors; Epithelial and Antennae type. Epithelial type receptors are found in animals, birds, reptiles, fishes etc. and are embedded and flushed with skin surface, usually in their nasal cavity. The general features are similar in them (15). The antennae type receptors are found in insects.

Consider the human olfactory cell as a particular case. The olfactory cell mainly has olfactory bipolar neurone from which cilia emerges. The secretions of Bowmann's glands keep the surface of epithelium moist and furnishes the necessary solvent. The odour molecule reaches the sensitive bipolar neurone after being dissolved in these secretions.. The continuous stream of secretions removes the remains of the stimulating substances and keeps the receptors ready for new odour molecules.

The general features of antennae type receptors of insects are similar. It contains naked dendrites bathed in biological fluid and are surrounded by cuticle having pores. The odour molecules reach sensitive dendritic membrane through these pores, after being dissolved in biological fluid (10).

2.02 Gustatory Receptors : These are groups of taste buds distributed usually on tongue. Each taste bud is a group of cells arranged in a small well around nerve endings. The molecules to be tasted reach these nerve endings by dissolving in water or saliva.

The above facts show that it is the nerve endings which receive odour or taste molecule (hereafter called guest molecule) dissolved in biological fluid. After the molecule has reached nerve endings the phenomena of odour and taste are indentical and further discussions apply equally to both these phenomena.

In case of taste phenomenon, it has been shown that it is not dependent on temperature in the range of 20°C to 30°C (near body temperature), [2] and the same is expected in case of odour phenomenon.

3.00 Molecule and nerve membrane : The guest molecule from air moveș to biological fluid medium in which the nerve endings are bathed and approaches towards the membrane of nerve endings (2.02). The condition of molecule in biological fluid is as described below.

3.01 Structural model of molecule in biological fluid: Since the biological fluids mainly contain water (nearly 80%), [9] a structural model must be formulated on the basis of structure of molecule in water.

The most accepted water hydrate model is being adopted [4]. The water molecules in liquid water exist in monomeric and polymeric forms, forming ice-like structure having cavities, which are capable of accommodating guest molecules. Two types of cavities of sizes 12.03 Å and 17.31 Å diameter have been suggested. The water molecules are hydrogen bonded with each other and have no free electron or proton to make further hydrogen bond. Dimers and trimers are very common.

Most of the gas hydrates constitute a class in which small molecules of many types occupy almost spherical holes in ice-like lattices made up of hydrogen bonded water molecules. The guest molecule will form such types of gas hydrates in water/biological fluid when dissolved in it. The guest molecle remains encaged in the firm enclosure made up of these H-bonded water molecules. The order of water molecules remains same. The guest and water molecules are not held up by any strong force and there is a remarkable degree of rotational mobility on the part of encaged molecule behaving as if in a vapour state [7]. The interaction of water and guest molecule appears primarily of van der Wall's type even for polar guest molecule. Guest molecule upto 5 Å in size fit easily into cages while a larger molecule, which can not be accommodated in a single cavity may occupy several adjacent sites.

The above model has been adopted for the interaction of guest molecule with nerve membrane. Before discussion of the interaction, some points regarding membrane and its excitation are discussed in the following para.

4.00 Membrane and excitation:

4.01 Membrane protein : It has been established that in the membrane, the lipid bilayer forms a barrier while its protein molecules after suffering conformational change are responsible for excitation. The exact mechanism, of excitation has not been understood. Thus, it is concluded that the guest molecule present at the outer side of nerve membrane, interacts with its protein, causing conformational change in it, resulting the excitation [11].

The globular protein of membrane contains 30% to 100% α-helix besides other parts. There is no water inside α-helix of protein. It has a delicately balanced structure. It performs highly specific and complicated chemical actions, which require high degree of organisation. In living organism it exists in native state. We have very little knowledge about its native or denatured state. The incorporation of certain proteins and polypeptides in artificial lipid bilayer may acquire selectivity, electrical excitability and even chemical receptor properties.

Protein molecules have two types of bonds; strong (Co-valent) and weak (Non-covalent) bonds. The weak hydrogen bonds are the predominent bonds governing folding and unfolding of protein.

Since no monopoles are observed inside the protein, the interactions of the multipoles are short and may be restricted to close neighbours only, in case of energy calculations.

Therefore, if the native protein is under stress, which increases continuously from zero stress (native state) to a higher value, the weak hydrogen bonds will break first resulting in its denatured state. This is a important conclusion for the present studies. Breaking of H-bond in protein of retina has been taken as primary cause of vision perception (14).

4.02 Model of membrane protein : A model of protein molecule has to be built up for further studies which should fulfil the excitation phenomenon. The main features of the proposed model are :

(a) Membrane excitation is due to conformational changes in the protein molecule.

(b) Structure, formation and stability of protein molecule depend mainly on non-covalent bondings.

(c) Hydrogen bonds are most important non-covalent bonds responsible for conversion of protein from native to denatured state which results in excitation in membrane.

Major portion of protein molecule comprises of right handed α-helix. It has been speculated that a reversible transition (from α-helix to sheet structure) may be the cause of muscle contraction and other types of movement in living organism. In a number of enzymes the biological activity depends on α-helical content (5). Thus it is concluded that :

(a) The globular protein is a delicately balanced structure and a small change in its energy may lead to a large change in its overall structure.

(b) Breaking of H-bond of α-helix, leading to conformational changes in protein molecule is the most probable cause of excitation.

(c) α-helix are present almost always on the surface of the protein [3].

On the basis of above facts, the following model has been adopted for further studies. In this model the polypeptide chain of protein is folded into α-helix. Th folding is maintained by H-bonds between -NH and =CO groups. Helix are assumed to be held rigidly with each other by di-sulphide bonds at some places (Fig. 1).

5.00 Interaction of guest molecule with protein : The Dipole Moment (D.M.) :

Interaction of guest molecule with protein molecule results in biological activity. Important facts regarding the interaction are as below:

(i) Guest molecules, under consideration may be taken as small molecule which neither react with water nor ionise in it e.g. O_2 CH_3, OH, $C_4 H_9 NH_2$ etc.

(ii) Protein molecule has α-helical content as a major portion bathed in water medium. It is inpenetrable to solvent and is surrounded by a firm layer of water of uni-molecular thickness [9].

(iii) The guest molecule when dissolved in water, lies in the interstitial cavity of water medium and forms a hydrate with it. The concentration of guest molecue is infinitly low ($< 10^{-3}$ M), [18] so that it can be considered as an isolated molecule, not affected by the presence of other similar molecules.

(iv) In a state of infinite dilution, the structure of guest molecule does not change and can be compared with its structure in the gaseous state [4]. There is very weak coupling between guest molecule and surrounding water molecules. At low concentrations any marked effects normally represent specific interactons of co-solute (guest molecule) with protein [13].

(v) The protein molecule consists of various bonds out of which H-bonds are weakest. Any small strain on these H-bonds may result in a major change in the protein molecule because it has a delicately balanced structure. Since denaturation process has been split into two step transition. Firstly native to a weekly co-operative transition and then to a state of denaturation (6).

We are interested in the strain produced on H-bond due to the presence of guest molecule which may result into complete danaturation process or a part of it as suggested above.

Other reasons for consideration of H-bonds of α-helix are:

1. α-helix are most abundent, delicately balanced structure with H-bonds continuously breaking and forming. They are found at the surface of protein.

?. Transition of α-helix to β-configuration may be responsible for biological activity which is the main consideration.

(vi) Now suppose a guest molecule 'M' when dissolved in water (or biological fluid), is approaching the helical part of protein molecule. When it is at the closest state to protein molecule, the situation is as shown in fig. 1. This type of bonding is reversible (8).

(vii) In the situation as depicted in para (vi), the stability of a H-bond due to its interaction with guest molecule 'M' (Say CH_3 OH) is to be investigated. A further Simplification gives the pictute (fig. 2) with only one H-bond and part of guest molecule in adjacent cavity. Here interactions except guest molecule and H-bond are neglected (17).

(viii) When there is no guest molecule 'M', the H-bond is stable with cage structure of water medium not occupied. The only difference between this situation and the situation depicted in Fig. 2 is that the cage structure in former is not occupied by a guest molecule, while in later case it is occupied.

(ix) There are two types of cages having exterior sizes equal to 12.03 Å and 17.31 Å giving an average exterior size as 14.68 Å. The average internal size is 4.2 Å.Thus the average thickness of cage is 3.14 Å.Take the distance of H-bond from unimolecular water layer as 1 Å. In case of polar guest molecule its polar part is most likely to be nearer to H-bond (Fig. 2 for CH_3 OH Molecule) and its minimum distance from H-bond will be little more than $1.52 + 3.14+1.0 = 5.66$ Å Say 6 Å. Thus the dipolar part of guest molecule can not come closer to H-bond than this value. If the molecule is bigger and can not be accommodated in one cavity, its polar part will lie nearer to H-bond and non-polar part will occupy adjacent cavities.

(x) When 'M' has entered the hydrate cage, it will interact electrostatically with H-bond.

(xi) The interaction energy between 'M' and H-bond is to be estimated. This is an additional energy due to presence of 'M' and will strain H-bond. If the interaction is of attractive in nature, the energy is surplus.

(xii) The interactions are due to di-or multipoles and therefore are short range interactions. This may be restricted to close neighbours only.

(xiii) The approximate maximum surplus strain energy experienced by H-bond due to presence of 'M' (Fig. 2) can be computed by Stockmeyer's potential. This energy,

$$E = 4\,\varepsilon\left[\left(\frac{\sigma}{\gamma}\right)^{6} - \left(\frac{\sigma}{\gamma}\right)^{12}\right] - \frac{2\,\mu_1\,\mu_2}{\gamma^3} \quad (\rightarrow\rightarrow)$$

(xiv) If the distance of separation 'r' between 'M' and H-bond is medium (6 Å to 15 Å) the 1st part of the equation (i) is quite low as compared to IInd part and energy E can be approximated to:

$E = \dfrac{2\mu_1\,\mu_2}{D^* r^3}$, where 'D*' is the dielectric constant.

(xv) D.M. of H-bond 'u' is a fixed quantity, thus

$$E\,\alpha\,\frac{\mu_2}{D^* r^3}$$

(xvi) If the protein molecule and medium remain same (as in case of nerve membrane) and value of 'r' is constant (in case of closest approach of polar part of 'M' to H-bond), the energy is dependent on 'μ_2' i..e. D.M. of guest molecule only.

(xvii) This (excess) energy will strain H-bond and may break it, if it is more than the bond energy of H-bond.

(xviii) Any strain to H-bond may result in drastic changes to the structure of α-helix which has a delicatly balanced structure. The Co-valent bonds of α-helix may act as an amplifier in magnifying the small changes in its structure into re-organisation of the molecule on a larger scale.

(xix) The dielectric constant 'D*', has been assumed to vary with 'r' in a linear manner with values as below :

r	o Å	6 Å	30 Å
D*	1	2	80

5.01 Hydrogen bond and interaction energy : Various investigators have given the following values of H-bond energy ; 0-2 K cal/mole, 100 cal/mole. A lower value of the bond energy has been suggested in aquous medium [6]. A value of 100 cal/mole seems to be a reasonable estimate.

As a typical case, interaction energy has been calculated for the case of interaction of CH_3 OH molecule with H-bond. The values of parameters are :-

CH_3 OH (molecule)		H-bond
value of 'ε'	0.82 K cal/mole	0.073 K cal/mole
value of 'σ "	3.69 Å	3.0 Å
value of 'μ'	1.7 D	1.3 D

Taking value of r=6 Å and dielectric constant D*=2, the total energy is nearly 90 cal/mole. The dipole-dipole interaction energy is 84% of total energy and this % increases with increase in value of 'r'.

Thus it is seen that the interaction energy is of the order of H-bond energy. It should also be kept in mind that the nature performs its functions in a more efficient manner and may need lesser energy than what has been estimated (see also para 5.00 v).

5.02 Conclusions : It has been seen that the guest molecule is capable of straining (or breaking) H-bonds, which brings a major change in protein of dendritic membrane, resulting in biological activity. The strain caused to H-bond is proportional to the Dipole Moment of guest molecule only (see para 5.00). Thus it is concluded that the Dipole Moment of guest molecule may be related directly to the biological activity of dendritic membrane of olfactory/gustatory receptors causing typical odour/taste.

With these ideas a plot of molecular Dipole Moment and its characteristic odour and taste (called odour/taste spectrum) has been proposed to check the validity of above conclusion.

6.00 THE SPECTRUM

6.01 Description of Odour and Taste: Odour of many molecules have been described by investigators in a variety of ways and it is little difficult to pin-point the correct statement. Therefore, a personal judgement has been applied in such cases in assigning a typical odour for these molecules. However, many molecules have been left out of this study where the odour description has not been precise e.g. 'sharp penetrating', 'mild odour', 'pungent' etc. Only those molecules have been selected where their odours have been described as typical odour e.g. 'rose odour', 'pear odour' etc.

Description of taste of molecules is even lesser in number and less descriptive. The procedure followed in case of odour molecule has been adopted in selecting the taste of a molecule.

6.02 Intensity : It has been felt that intensity of odour/taste depends upon the rate of molecules reaching the membrane of receptors, which depends upon the physical parameters such as wind velocity, temperature, rate of out flow of molecules from the source, its water solubility etc. It has been observed that odour/taste of a molecule changes due to wide variations in its intensity. This may be due to some different changes in membrane protein because of presence of molecules in higher number than

what is considered here i.e. in infinitly low number. In present studies this factor of higher intensity has not been considered.

6.03 Odour/Taste Molecule: Most of the molecules under consideration exist in monomeric forms e.g. halo-forms, highly halogenated compounds, acytylenes, ketones, ethers, esters, olefines, aromatic and tertiary amines, nitriles, isonitriles, saturated hydrocarbon, carbon di-sulfide etc.

There are molecules which self associate to give a bigger molecule e.g. phenols, alcohols, inorganic and carboxylic acids, primary and secondary amines etc. Even these molecules exist in monomeric forms if concentration is less than 10^{-3} M.

The molecules falling in above categories have been considered in this study. Molecules reacting in atmosphere or reactng with membrane components have been excluded from these studies.

6.04 Value of Dipole Moment : Due to following reasons difficulty has been felt in selecting the value of D.M. of a particular molecule.

(a) These values have been reported for gas form as well as in solutions of various solvents at different temperatures. The solvent effect of non-polar solute has been found to be negligible (12). However, gas values near body temperature has been selected as most appropriate value.

(b) In many cases very few values are available and it is difficult to assess the correctness of the reported value (s). Such values have been used with some restraint.

6.05 Odour/Taste Spectrum: On the basis of conclusion drawn in para 5.02 that the **molecular dipole moment is directly related to its characteristic odour/taste;** a spectrum of Dipole Moment expressed in Debye unit 'D' and the characteristic odour/taste of the molecules has been proposed (See Annexure). Its general features are:

(i) The plot ranges from D.M. = 0 to 3.86 D

(ii) The spectrum is applicable in case of odour/taste molecules present in low concentrations.

(iii) There are 113 distinct odours and 48 distinct tastes.

(iv) Molecules with D.M. < 0.15 D are odour/tasteless. This indicates that the interaction energy in such cases is not enough to cause excitation to the extent that it is perceptible to human organs.

(v) The spectrum is broadly divided into following parts.

D.M. = 0 to 0.15 D-odour/tasteless
= 0.15 D to 0.4 D-Hydrocarbon odour/taste
= 0.4 D to 0.65 D-Oily odour/taste
= 0.65 D to 1.75 D-Mixed odour/taste
= 1.75 D to 1.95 D-Fruity, Flowery odour/taste (values close to water)
= 1.95 D to 2.40 D-Animal odour/taste
= 2.40 D to 3.56 D-Perfumery odour/taste.

(vi) Water occupies a special status. It is odour/tasteless even though its D.M. value is very high. It is because when water molecule reaches near membrane which is already bathed in water medium, the situation near it does not change at all. Therefore, water is odour/tasteless.

7.00 Conclusions : The various conclusions drawn from present studies are listed below:

(i) The excitation in olfactory/gustatory organ is due to interaction of odour/taste molecule with its sensitive membrane.

(ii) Molecules having same D.M. have similar odour/taste, while molecules of different D.M. have different odour/taste. This verifies the inference drawn in para 5.02 that D.M. is directly related to the biological activity of odour/taste molecules.

(iii) Same value of D.M. is representative of both odour and taste of the molecule simultaneously e.g. D.M. = 1.44 D corresponds to acetic acid odour as well as acetic acid taste.

(iv) The molecules of a group of hydrocarbon having same polar part but different chain length have similar odour/taste. It is because the polar part is nearer to H-bond of membrane protein while the tail is away and the effect of tail is very less as compared to its polar part.

(v) The resolution power of human (Olfactory/Gustatory) receptor membrane in terms of D.M. is 0.01 D i.e. the membrane in both cases is capable of distinguishing molecules if their D.M. differ by more than 0.01 D.

(vi) The general conclusions drawn from studies of human receptors seems to apply in case of receptors of other vertebrates, insects, birds etc.. because there seems to be no reasons contradictory to it in light of para 2.00. Here it is emphasized that human olfactory sense is quite inferior to many animals and insects.

(vii) Pleasent or unpleasentness depends upon individual nature. Normally all odour/taste, felt at low concentrations are pleasent and are unpleasent at high concentrations.

(viii) The spectrum will help in predicting odour and taste of a molecule of known value of D.M. and vice-versa. Knowledge of D.M. will help in finalising molecular structure of a molecule.

(ix) The spectrum will be helpful in finding out new synthetic flavours.

ANNEXURF

Odour/Taste Spectrum O = Odour

T = Taste

D.M. (D)	Odour/Taste	D.M. (D)	Odour/taste.
0.00	Odourless/Tasteless	0.85	Iodoform (O)
0.10	Odourless/Tasteless	0.87	Chloral (O)
0.15	Nitrous Oxide (O)	0.90	Rotten egg (O)
0.20		0.95	Chloroform (O, T)
0.25		1.00	Grapes (O, T)
0.30		1.01	Orange blossom (O)
0.35	Hydrocarbon (O)	1.02	
0.40	Kerosine (O)	1.03	Metallic (O)
0.45	Tarpene (O)	1.04	
0.475	Benzene (O)	1.05	Gerfanium (O)
0.50	Fried Potato (O..T.)	1.06	1.68
0.55		1.065	Napthalene (O)
0.56	Ozone (O)	1.08	Fishy (O)
0.60		1.10	Halogen (O)
0.65		1.11	
0.70		1.12	
0.75		1.13	Chloroethynyl Benzene (O)
0.80		1.14	

1.15		1.46	
1.16	Diethyl ether (O)	1.48	
1.17	Sweet (T)	1.49	Ammonia (O)
1.18	1.805	1.50	
1.19	Caproic (O, T)	1.51	Cabbage/radish (O,T)
1.20		1.52	
1.21		1.53	Sulfuraceous/Shunk (0)
1.22		1.54	
1.23	Piperidine (O)	1.55	Thyme (O)
1.24		1.56	
1.25	Anisole (O)	1.565	Peppery (O)
1.26		1.575	Woody (O, T)
1.28		1.58	
1.30	Marigold/wall flower (O)	1.585	Dry tea leaves (O)
1.32		1.595	Ylang-Ylang (O)
1.34	Hyacinth (O)	1.60	Narcissus (O)
1.36	Earthy (O)	1.605	Lily (O)
1.37	Hazelnut (O, T)	1.61	Jasmine (O)
1.38		1.62	Orrissy (O)
1.40	Rancid Buttery Cheesy (O, T)	1.64	Dry leather (O)
1.42		1.66	Phenolic/medicinal (O, T)
1.44	Acetic acid (O,T)	1.67	Chemical (O, T)

1.685	Fusal oil (O)	1.89	Anisic (O)
1.70	Alcohlic (O, T)	1.88	Strawberry (O, T)
1.72	Mandrain (Citrusy) (O, T)	1.90	
1.73	Lemon (O, T)	1.91	Oily burnt (O.T.)
1.74	Orange (O)	1.92	Rosemerry/Lavandin
1.76	Waxy (O)	1.93	Piney (O)
1.77	Honey (O, T.)	1.94	
1.78	Rose (O, T)	1.96	
1.79		1.98	
1.80	Apple (O,T) Pear (O, T)	1.99	Onion (O.T.)
1.81	Banana (O, T)	2.00	Garlic (O.T.)
1.815	Pineapple (O, T)	2.02	
1.82	Apricot (O, T)	2.04	
1.83	Blackcurrent (O, T)	2.06	
1.835	Water (odour & Tasteless) (O, T)	2.07	Orange leaf/flower (O)
1.84		2.08	
1.845	Ashtray (O)	2.12	
1.85		2.14	Animal (O, T)
1.86	Brandy (O)	2.155	Indolic (O)
1.87	Rum (O, T)	2.16	

2.17	Tarry (O)	2.50	
2.18		2.51	Wintergreen (O.T.)
2.19	Nitric acid (O)	2.52	
2.20		2.54	
2.21	Burnt/smoky (O, T)	2.55	Clove Oil (O.T.)
2.22		2.56	
2.24	Buttery (O, T)	2.58	Peach (O.T.)
2.25	Coconut (O, T)	2.60	
2.26		2.62	
2.28	Earthy (O)	2.63	Musty (O)
2.30	Oily (O)	2.64	
2.32	Walnuts (O)	2.65	Gassy (O)
2.34	Soapy (O)	2.66	
2.36		2.67	Tobacco leaves (O)
2.37	1, 5 di-chloropentane (O)	2.68	
2.38		2.70	
2.40		2.72	Roasted coffee (O)
2.42		2.74	
2.44		2.75	Bitter almond (O.T.)
2.45	Vanilla (O.T.)	2.76	
2.46	Chocolate (O)	2.78	Musky (O.T.)
2.475	Winey (O)	2.80	Mushroom (O)
2.48		2.82	

2.84			
2.85	Camphor (O)	3.16	Hawthorn (O)
2.86		3.18	
2.88	Peppermint (O.T.)	3.20	
2.90		3.22	
2.91	Acetone (O.T.)	3.24	
2.92		3.26	
2.94		3.28	
2.95	Balsamic (O)	3.30	Hay (O)
2.96		3.32	
2.98	Burning fat (O)	3.34	
3.00		3.36	
3.01		3.38	
3.02		3.40	
3.03		3.42	
3.04		3.44	
3.05	Mimosa (O)	3.45	Cederwood (O.T.)
3.06		3.46	
3.07		3.48	
3.08		3.50	
3.09		3.52	
3.10	New mown hay (O)	3.54	Heliotropine (O.T.)
3 12		3.56	

3.14		3.58	Benjoin (O)
3.60		3.80	Caramellic (O.T)
3.62		3.82	
3.64		3.84	
3.66	Bread (O.T.)	3.86	Orris/Ambre/vetiver (O)
3.68		3.88	
3.70	Cinnamon (O.T.)	3.90	
3.72		3.92	
3.74		3.94	
3.75	Spicy (O.T.)	3.96	
3.76		3.98	
3.78		4.00	

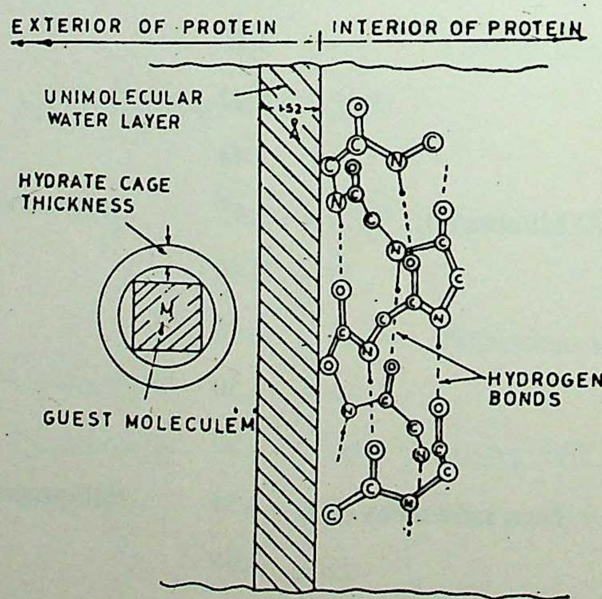

FIG.-1 HELIX AND GUEST MOLECULE

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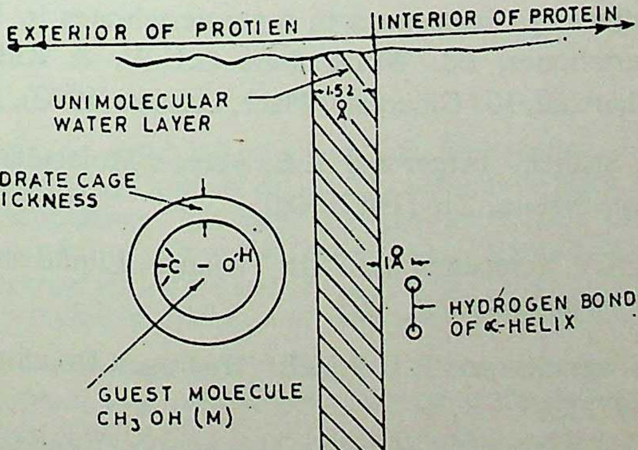

FIG-2 HYDROGEN BOND AND GUEST MOLECULE

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 3 (1989), 33-38

A STUDY OF ESSENTIAL OIL FROM ERIGERON LINIFOLIUS WILLD

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ABSTRACT

Essential oil from *Erigeron linifolius* willd, a widely distributed plant in Kumaon and Garhwal regions, possessed

$$d^4_{20°} = 0.8782 \; ; n^D_{20°} = 1.53 \; ; \alpha^D_{20°} = 41° 21'$$

Acid value = 0.3 ; Ester value = 50.2 and Carbonyl value = 0.8. GLC study of the oil using Honeywell HP 7620–A model showed the presence of α-pinene, β-pinene, myrcene, ocimene, limonene, caryophyllene, humulene, cubebne and lachrophyllum ester.

Key words and phrases : Essential oil, *Erigeron linifolius Willd.*
Chemistry-subject classification : Pharmaceutical Analysis Essential oils-62.

Erigeron linifolius Willd, belonging to N.O. Compositae, is widely distributed in Kumaon and Garhwal regions. Leaves of the plant are used in Malaya for rheumatism, lumbago and poulticing [1]. Chemical composition of essential oils from *Erigeron annus, Erigeron karwinsky anus* DC, and *Erigeron philadiphicus* [2] [3] [4] have been reported but very little work is reported on essential oil of *Erigeron linifolius* Willd.

Experimental : The essential oil was obtained by water-steam distillation of the plant in a copper still, extracting the distillate with petroleum ether (b 60-80°), drying over sodium sulphate and concentrating

the organic phase under reduced pressure (yield = 0.2%). Physico-chemical properties of the oil were :

Sp.gravity/$_{20}$°=0.8782 ; Ref. index/$_{20}$° = 1.53 ; Sp. rotation/$_{20}$° = + 41°21'

Acid value = 0.3 ; Ester value = 50.2 and Carbonyl value as $C_{10}H_{16}O$ (oximation) = 0.8.

Fine needle shaped crystals formed, on keeping the oil overnight during winters, were separated. The process was repeated till formation of crystals ceased. Hydrocarbon and Oxygenated fractions of the solid free oil were separated by using a column packed with silicic acid and n-hexane as eluting agents [5]. The two fractions were subjected to GLC examination, separately, under similar parameters. Different constituents of the fractions were identified by injection method and comparison of retention times under the same operating parameters using the pure compounds. The conditions of the GLC analysis were :

Column	..	Carbowax 20 M, 10%.
Carrier gas	..	Nitrogen (30ml per minute)
Column temperature	..	200°
Injection temperature	..	260°
Detector type	..	Thermal conductivity detector (temperature 300°)
Chart speed	..	0.25" per minute.
Sensitivity	..	10
Chromatograph model	..	Honeywell HP 7620-A.

Results of GLC of Hydrocarbon fraction (F1) and Oxygenated fraction (F2) are summarised in table 1 and table 2, respectively.

The solid fraction, separated from the oil earlier was further purified by column chromatography and TLC. Solvent systems used were chloroform, petroleum ether-pyridene (95:5). After development the plates were sprayed with fluorescein-bromine reagent for visual identification. Better resolution was obtained with petroleum ether pyridene. Two

fractions, isolated, in pure form, were identified as cis-lachrophyllum ester and matricaria ester by Co-TLC and spectroscopic examination.

Spectroscopic examination of the fractions :

Fraction 1:

The fraction melted at 32°C and possessed molecular formula $C_{11}H_{10}O_2$ (C=75.8%, H=5.74% and 0=18.46%, Molecular weight = 174).

UV : λ max at 218 and 308 nm indicated the presence of symmetrical diene and three conjugated system units, respectively

IR : bands at 3650 cm^{-1} –3100 cm^{-1} (–CH–stretching); 2800 cm^{-1} 3000 cm^{-1} (CH_3–CH_2–$CH_2C{\equiv}C$–); 1600 cm^{-1} (strong) for carboxylate ion; 1750 cm^{-1} for α–β–unsaturated ester; 2200 cm^{-1} (medium) for –C ≡ C–; 970 cm^{-1} for –CH≡CH– and 1600 cm^{-1} for symmetrical diene.

IR spectrum of the compound was found the finger print of IR spectrum of matricaria ester.

Fraction 2 : The fraction melted at 33°C and possessed molecular forumula $C_{11}H_{12}O_2$ (C = 75.0%, H = 6.8% and 0 = 18.2%. Molecular weight = 176).

UV : λ max at 219 nm (for –C≡C–C≡C– unit) and 275 nm (for two conjugated system unit).

IR : bands at 3650 cm^{-1} –3100 cm^{-1} (–CH stretching); 2800cm^1– 3000 cm^{-1} (CH_3–CH_2–CH_2–C≡C–) ; 1600 cm^{-1} (strong) for carboxylate ion; 1445 cm^{-1} (medium) for –CH_2–C≡C– ; 1400 cm^{-1} (weak) for carboxylate ion; 1750 cm^{-1} for α–β–unsaturated ester; 2200 cm^{-1} (medium) for –C≡C– ;970 cm^{-1} for –CH=CH– and 1600 cm–1 for symmetrical diene.

IR spectrum of isolated lachrophyllum ester was found the finger print of IR of the pure sample of the ester.

PMR : A triplet at 1.0 (δ) for $-CH_3$; a hextet at 1.57 (δ) for $-CH_2-$ $(CH_2-CH_2-CH_2-)$; at 6.13 (δ) for protons of $>C-C<^{COOR}_{H}$ and a singlet at 3.75 (δ) for $-OCH_3$.

Combustion analysis, UV, IR and PMR results correspond to the following structure :

$(CH_3-CH_3-CH_2-(C{\equiv}C-)_2\ CH{=}CH-COOCH_3$

The oil was also studied for antibacterial and antifungal properties by 'Filter paper disc diffusion technique' following Difco Laboratories Manual (6). The oil showed inactivity against *E. coli* and *Tricophyton mentagrophytes* . It showed slight activity against *S. Aureus*.

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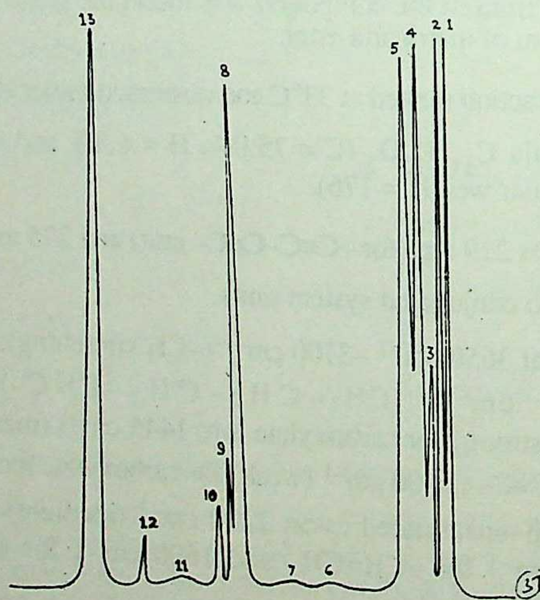


F 1 : GLC OF HYDROCARBON FRACTION

TABLE - 1

GLC of Hydrocarbon fraction

Peak No.	Compound identified	Mode of identification
1.	α-pinene	IR, T_R
2.	β-pinene	IR, T_R
3.	myrcene	IR, T_R
4.	ocimene	IR, T_R
5.	limonene	IR, T_R
6.	u.i.	
7.	u.i.	
8.	caryophyllene	IR, T_R
9.	u.i.	
10.	humulene	IR, T_R
11.	u.i.	
12.	u.i.	
13.	cubebne	IR, T_R

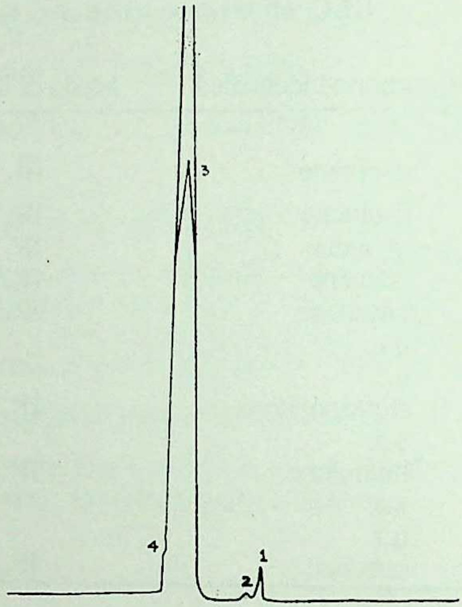
In this table u.i. denotes unidentified peaks.
IR denotes Infra red spectroscopy.
T_R denotes relative retention times.

TABLE - 2

GLC of Oxygenated fraction

Peak No.	Compound identified	Mode of identification
1.	u. i.	
2.	u. i.	
3.	lachrophyllum ester	IR, T_R
4.	u. i.	

In this table u.i. denotes unidentified peaks.
IR denotes Infra red spectroscopy.
T_R denotes relative retention times.



F 2: GLC OF OXYGENATED FRACTION

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 3 (1989), 39-48

ON CONVEXITY IN CONVEX METRIC SPACES WITH APPLICATION

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1. ABSTRACT

In this paper we prove a generalization of a fixed point theorem from [7] for Takahashi convex metric spaces. For this class a characterization of convex hull is given too.

Key words and phrases: Convex metric space, fixed point. AMS (MOS) Subject Classifications (1980) : 47H10.

2. INTRODUCTION

In 1970 Takahashi [5] introduced the definition of convexity in metric space and generalized some important fixed point theorems previously proved for Banach spaces. Subsequently, Machado [3], Talman [6], Naimpally and Singh [4], Guay and Singh [1], Hadžiac′ and Gajic′ [2], among others have obtained additional results in this setting.

This paper is a continuation of these investigations. Without additional conditions on convex structure we are able to give a characterization of convex hull. The major fact here is that the diameter is invariant under passage to so defined convex hull. In second part, for Takahashi convex metric spaces, we prove a generalization of a fixed point theorem from [7].

3. PRELIMINARIES

DEFINITION 1. Let X be a metric space and I be the closed unit interval. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X, \lambda \in I$,

$$d(u, W(x,y,\lambda)) \leq \lambda\, d(u,x) + (I-\lambda)\, d(u, y), \text{ for all } u \in X.$$

X together with a convex structure is called a convex metric space. Clearly any convex subset of a normed space is a convex metric space with $W(x, y, \lambda) = \lambda x + (I-\lambda) y$.

DEFINITION 2. Let X be a convex metric space. A nonempty subset K of X is said to be convex if and only if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.

Takahashi has shown ([15]) that open and closed balls are convex, and that the arbitrary intersection of convex sets is convex too.

For arbitrary $A \subset X$ let

(1) $$\widetilde{W}(A) := \{W(x,y,\lambda) : x,y \in A, \lambda \in I\}.$$

It's easy to see that $\widetilde{W} : P(X) \to P(X)$ is a mapping with properties:

i) $A \subset \widetilde{W}(A)$, for $A \subset X$;

ii) $A \subset B$ implies $\widetilde{W}(A) \subset \widetilde{W}(B)$, for $A, B \in P(X)$

iii) $\widetilde{W}(A \cap B) \subset \widetilde{W}(A) \cap \widetilde{W}(B)$, for any $A, B \in P(X)$.

Using this notation we can say that K is convex iff $\widetilde{W}(K) \subset K$.

A few additional definitions will be needed too.

DEFINITION 3. A convex metric space X will be said to have Property (C) iff every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.

Remark. Every weakly compact convex subset of a Banach space has Property (C).

DEFINITION 4. Let X be a convex metric space and A be a nonempty, closed, convex, bounded set in X. For $x \in X$ we set

$$r_X(A) = \sup_{y \in A} d(x, y),$$

$$r(A) = \inf_{x \in A} r_X(A).$$

We then define $A_C = \{x \in A : r_X(A) = r(A)\}$ to be the centre of A. We denote the diameter of a subset A of X by

$$\delta(A) = \sup\{d(x, y) : x, y \in A\}.$$

DEFINITION 5. A point $x \in A$ is a diametral point of A iff $\sup_{y \in A} d(x,y) = \delta(A)$.

DEFINITION 6. A convex metric space X is said to have normal structure iff for each closed bounded convex subset A of X, which contains at least two points, there exists $x \in A$ which is not a diametral point for A.

Remark. it is clear that any compact convex metric space has normal structure.

4. RESULTS

Now, let us recall:

DEFINITION 7. The convex hull of a set A ($A \subset X$) is the intersection of all convex sets in X containing A and it is denoted by conv A.

It is obvious that if A is a convex set then $\widetilde{W}^n(A) = \widetilde{W}(\widetilde{W}(\ldots \widetilde{W}(A) \ldots) \subset A$ for any $n \in N$.

If we set :

$$A_n = \widetilde{W}^n(A), \qquad (A \subset X),$$

the sequence $\{A_n\}_{n\in N}$ is increasing so lim sup A_n exists and

$$\limsup A_n = \liminf A_n = \lim A_n = \bigcup_{n=1}^{\infty} A_n.$$

PROPOSITION 1. Let X be a convex metric space. Then

$$\text{(2)} \qquad \operatorname{conv} A = \lim A_n = \bigcup_{n=1}^{\infty} A_n, \qquad (A \subset X),$$

PROOF. If $x, y \in \bigcup_{n=1}^{\infty} A_n$, then there exists a positive integer n_0 (say) such that $x, y \in A_{n_0}$. So, for $\lambda \in I$, $W(x, y, \lambda) \in A_{n_0+1} \subset \bigcup_{n=1}^{\infty} A_n$. Thus $\bigcup_{n=1}^{\infty} A_n$ is convex and contains A. Further for any convex set C containing A, $\widetilde{W}^n(A) \subset C$ for every $n \in N$, i.e., $\bigcup_{n=1}^{\infty} A_n \subset C$. So, $\bigcup_{n=1}^{\infty} A_n = \operatorname{conv} A$.

In the remainder of the paper (X, d) will denote a convex metric space.

PROPOSITION 2. For any subset A of (X, d)

$\delta(\operatorname{conv} A) = \delta(A)$.

PROOF. Since $A \subset \operatorname{conv} A$ then $\delta(A) \leq \delta(\operatorname{conv} A)$. Now, let x and y be in conv A. If $x, y \in A$ it is obvious that $d(x,y) \leq \delta(A)$, so let one of them, for instance x, be in conv $A \backslash A$ and the second one from A.

Since $x \in$ conv A there exists $n_0 \in N$ such that $x \in \widetilde{W}^{n_0}(A)$. But it means that there exist $x_1, x_2 \in \widetilde{W}^{n_0-1}(A)$, $\lambda_1 \in [0, 1]$ so that $x = W(x_1, x_2, \lambda_1)$ and then :

$$d(x, y) = d(W(x_1, x_2, \lambda_1), y) \leq \lambda_1 d(x_1, y) + (1 - \lambda_1) d(x_2, y).$$

By induction one can see that there exist subset $\{\tilde{x}_1\}_{i \in I} \subset A$ (I finite set) and $\{\alpha_i\}_{i \in I,}$ $\alpha_i \geq 0, \Sigma \alpha_i = 1,$ such that

$$d(x, y) \leq \sum_{i \in I} \alpha_i d(\tilde{x}_i, y).$$

Since

$$d(\tilde{x}_i, y) \leq \delta(A) \qquad \text{for } i \in I$$

we prove that :

(3) $$d(x, y) \leq \delta(A).$$

Similarly one can prove that (3) is valid even in the case when $y \in$ conv A.

5. A FIXED POINT THEOREM

THEOREM. Let (X,d) be a metric space with continuous convex structure and let K be a closed convex bounded subset of (X, d) with normal structure and Property (C).

If $A: K \to K$ is a continuous mapping such that for $x,y \in K$

(4) $$d(Ax, Ay) \leq max\{d(x,y), d(x, Ax), d(y, Ay)\ d(x, Ay), d(y, Ax)\}$$

then A has a fixed point.

PROOF, Let F be the family of all nonempty closed convex subset $F \subset K$ so that $A(F) \subset F$. Then F is nonempty since $K \in F$. Partilally order F by inclusion. Let $S = \{F_i\}_{i \in \Delta}$ be a decreasing chain in F.

From Property (C) it follows that $F_o = \bigcap_{i \in I} F_i \neq \phi$. So $F_0 \in F$.

Thus any chain in F has a greatest lower bound and by Zorn's Lemma there is a minimal member F in F. We claim that F is a singleton

set. If not, then as shown by Takahashi [5] the center of F, denoted by F_C, is nonempt proper closed convex subset of F. It's easy to see that :

$$\delta(F_C) \le r(F) \le \delta(F).$$

Let us define a sequence :

$F_O = F_C$ and

$F_{k+1} = conv\,(F_k \cup A\,(F_k)),\ k = 0, 1, \ldots\ldots\ldots$

Obviously $F_k \subset F_{k+1}$, $(k = 0, 1, 2, \ldots..)$ we shall prove by induction that :

(5) $\delta_k = \delta(F_k) \le r\,(F) = r$ for any $k \in N$.

For k=0 (5) is valid. Suppose that it is valid for k=0,1,2,...,....,m and let us prove that (5) is valid for $k = m+1$ too.

By definition of δ(F) for any sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$ $(n \in N)$, $\lim_{n \to \infty} \varepsilon_n = 0$ there exist $\tilde{x}_n, \tilde{y}_n \,\varepsilon F_{m+1}$ so that

$$\delta_{m+1} - \varepsilon_n \le d\,(\tilde{x}_n, \tilde{y}_n).$$

By Proposition 2 it is sufficient to analyse next three cases :

i) $\tilde{x}_n, \tilde{y}_n \,\varepsilon F_m$ $(n = 1, 2, \ldots)$,

ii) $\tilde{x}_n = x_n, \tilde{y}_n = Ay_n, (x_n, y_n, \varepsilon F_m, n = 1, 2, ..)$

iii) $\tilde{x}_n = Ax_n, \tilde{y}_n = Ay_n, (x_n, y_n, \varepsilon F_m, n = 1, 2, ..)$

At first case it is obvious that $\delta_{m+1} \le$ r. So let us see the second one. For any $x \in F_O$ we have that:

(6) $d\,(x, Ax) \le r$.

We shall suppose that (6) is valid for $x \,\varepsilon F_k$ $\{k = 0, 1, \ldots m-1\}$ and prove that it is valid for $k=m$.

For any $x \in F_m$, by Proposition 1,

$$x \in \widetilde{W}^{n_o}\left(F_{m-1} \cup A\left(F_{m-1}\right)\right)$$

for some $n_o \in N$.

Then

$$d(x, Ax) \leq \sum_{j \in I_1} \gamma_j d\left(x_j, Ax\right) + \sum_{j \in I_2} \gamma_j d\left(Ax_j, Ax\right), \tag{7}$$

for $x_j \in F_{m-1}, j \in I = I_1 \cup I_2$, ($I$–finite set) $I_1 \cap I_2 = \phi$

$$\sum_{j \in I} \gamma_j = 1, \gamma_j \geq 0 \text{ for } j \in I.$$

As in [7] it is sufficient to look only for the case when $\sum_{j \in I_1} \gamma_j \neq 0$. Further we have that :

$$d(x, Ax) \leq \sum_{j \in I_1} \gamma_j d\left(x_j, Ax\right) + \sum_{j \in I_2^{(1)}} \gamma_j d\left(x_j, x\right) + \sum_{j \in I_2^{(2)}} \gamma_j d(x, Ax)$$

$$+ \sum_{j \in I_2^{(3)}} \gamma_j d\left(x_j, Ax_j\right) + \sum_{j \in I_2^{(4)}} \gamma_j d\left(x, Ax_j\right) + \sum_{j \in I_2^{(5)}} \gamma_j d\left(x_j, Ax\right)$$

where we suppose

for $i \in I_2^{(1)}$ *that* $d(Ax_j, Ax) \leq d(x_j, x)$;

for $i \in I_2^{(2)}$ that $d(Ax_j, Ax) \leq d(x, Ax)$;

for $i \in I_2^{(3)}$ that $d(Ax_j, Ax) \leq d(x_j, Ax_j)$;

for $i \in I_2^{(4)}$ that $d(Ax_j, Ax) \leq d(x, Ax_j)$;

for $i \in _2(5)$ that $d(Ax_j, Ax) \leq d(x_j, Ax)$;

and $I_2^{(j)} \cap I_2^{(k)} = \phi$ *for* $k \neq j$.

Now using the hypothesis one can see that

$$d(x, Ax) \leq \sum_{j \in I_1} \gamma_j d\left(x_j, Ax\right) + r \sum_{j \in I_2^{(1)} \cup I_2^{(3)}} \gamma_j + \sum_{j \in I_2^{(2)}} \gamma_j d(x, Ax)$$

$$+ \sum_{j\in I_2^{(4)}} \gamma_j d(x, Ax_j) + \sum_{j\in I_2^{(5)}} \gamma_j d(x_j, Ax).$$

Since, by induction simillarly we have that

$$d(x, Ax_j) \le \sum_{k\in J_j^{(1)}} \beta_k d(\hat{x}_k, x_j) + \sum_{k\in J_j^{(2)}} \beta_k d(x_k, Ax_j)$$

$$+ \sum_{k\in J_j^{(3)}} \beta_k d(\hat{x}_k, A\hat{x}_k) + \sum_{k\in J_j^{(4)}} \beta_k d(x_j, A\hat{x}_k)$$

$$+ \sum_{k\in J_j^{(5)}} \beta_k d(\hat{x}_k, A\hat{x}_j)$$

for $\hat{x}_k \in F_{m-1}, k\left(\in J_j = \bigcup_{i=1}^{5} J_j^{(i)}, \sum_{k\in J_j} \beta_k = 1, \beta_k \ge 0, k\in J_j, \sum_{k\in J_j^{(1)}} \beta_k \ne 0\right).$

So

$$d(x, Ax_j) \le r \text{ and}$$

$$d(x, Ax) \le \sum_{j\in I_1} \gamma_j d(x_j, Ax) + \left(\sum_{j\in I_2^{(1)}} + \sum_{j\in I_2^{(3)}} + \sum_{j\in I_2^{(4)}}\right) g$$

$$(8) \quad + \sum_{j\in I_2^{(2)}} \gamma_j d(x, Ax) + \sum_{j\in I_2^{(5)}} \gamma_j d(x_j, Ax)$$

After not more than n_0- steps we shall have that :

$$d(x, Ax) \le \sum_{j\in I^*} \gamma_j^* d(v_j, Ax) + \gamma_o^* r, \quad for\ \gamma_i^* \ge 0 \ i\in \{0\} \cup I^*$$

$\gamma_0^* + \sum_{j\in I^*} \gamma_j^* = 1$ *and* $v_j \in F_0, j\in I^*$. Since F_0 is centre we have that

$$d(v_j, Ax) \le r$$

which implies that :

$$d(x, Ax) \le r \qquad \text{for all } x\in F_m.$$

Similarly one can prove that $d(x, Ay) \leq r$ for all $x, y \in F_m$

so in the second case we have

$$\delta_{m+1} - \varepsilon_n \leq d(\bar{x}_n, \bar{y}_n) = d(x_n, Ay_n) \leq r, n \in N$$

and consequently

$$\delta_{m+1} \leq r.$$

Using (4) it is easy to prove this inequality in the case iii). So for any $m \in N$

$$\delta_m \leq r.$$

Let us define $F_\infty := \bigcup_{k=0}^{\infty} F_k$

F_0 is nonemty set. So F_∞ is nonempty too.

Since $\delta(F_\infty) < r < \delta(F)$, F_∞ is closed proper subset of F. Moreover, W is continuous and this guarantees that the closure of convex set is convex. Further, mapping A is continuous so:

$$A(F_\infty) \subset F_\infty$$

Therefore $F_\infty \subset F$, a contradiction to the minimality of F. Hence F consists of a single element which is a fixed point for A.

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THERMAL STRESS IN A SEMI-INFINITE ROD OF NONHOMOGNEOUS MATERIAL

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ABSTRACT

Distribution of thermal stress in a semi-infinite elastic rod of nonhomogeneous material has been found. The density and the Young's modulus of the material are supposed to be functions of position. A constant temperature has been maintained at the end of the rod. Distribution of stress has been shown graphically for particular values of the parameters involved.

Key words : Thermal Stress, Young's modulus, Nonhomogeneous material, Laplace transform.

INTRODUCTION

The problem of determining thermal stresses in a elastic body has been gaining increasing importance because there appear many practical problems where mechanical structures are subjected to high temperature and pressure. Barrekette [1], Lal [3] and Nowinski [4] considered statical problems. But due to complexity involved in the governing differential equations it becomes quite difficult to solve thermoelastic problems when

time appears as a variable. The problem of determining stress in rods is usually based on a one-dimensional stress-strain law and the hypothesis that plane sections of the rod remain plane after heating. The dynamical thermoelastic problem had been solved by Roychaudhuri [6] where he found the distribution of thermal stress in a thin rod of finite length one end of which was maintained at a constant temperature, the material of the rod being homogeneous. As regards nonhomogeneous problems, the governing differential equations become little more complicated and as a result sometimes simplifying assumptions are adopted in order to avoid such complications. The technique of Laplace transform is used, but due to natural difficulties in inversion, asymptotic expansion is followed to get solution of the problem valid for small time or limiting values of parameters are assumed. Latter type of non-homogeneous problem has been considered by Pal [5]. In the present investigation an attempt has been made to find thermoelastic stress distribution in a semi-infinite rod the end of which has been maintained at a constant temperature. The material of the rod has been assumed nonhomogeneous in the sense that both young's modulus E and the density ρ of the material are functions of position.

FORMULATION OF THE PROBLEM

Choosing x-axis along the axis of the rod let the position of the rod be given by $x \geq 0$. The differential equation of motion is

$$\frac{\partial \sigma}{\partial x}=\rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots (1)$$

where $\rho = \rho(x)$ is the density of the material and $u = u(x,t)$ is the longitudinal displacement of the centroid of the cross-section, which in the equilibrium state is at a distance x from the origin. The $\sigma = \sigma(x,t)$ is given by

$$\sigma = E\left(\frac{\partial u}{\partial x} - \alpha T\right) \qquad \ldots (2)$$

where $E = E(x)$ is the Young's modulus, $T = T(x,t)$ is the temperature at time t at α distance x from the end and α is the coefficient of linear expansion of the material of the rod.

Let the nonhomogeneity of the material be characterised by

$$E = E_o e^{-az} \qquad \text{... (3)}$$

$$\rho = \rho_o e^{-az}, \ a \neq o$$

where $z = x/L$ ... (4)
and L has the dimension of length.

In view of (3), substitution of (2) into (1) leads to

$$\frac{\partial^2 u}{\partial z^2} - a\frac{\partial u}{\partial z} - \frac{\partial^2 u}{\partial \tau^2} = \alpha L \left(\frac{\partial T}{\partial z} - aT\right) \qquad \text{... (5)}$$

$$\text{where } \tau = \frac{ct}{L},\ c^2 = E_o/\rho_o. \qquad \text{... (6)}$$

The differential equation for the determination of temperature $T(x,t)$ at any point x at time t is

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2},\ x > 0, t > 0 \qquad \text{... (7)}$$

The quantity k generally depends upon the thermal conductivity, specific heat and the density of the material. For the sake of simplicity in the solution of (7) we assume that k is a constant.

Using (4) and (6), (7) becomes

$$\frac{\partial T}{\partial t} = h\frac{\partial^2 T}{\partial z^2} \qquad \text{... (8)}$$

Where $h = k/cL$.

The initial and boundary conditions of the problem are

$$u(z, 0) = 0 = \frac{\partial}{\partial \tau} u(z, 0),\ z > 0 \qquad \text{... (9)}$$

$$T(z, 0) = 0 \quad \dots (10)$$

$$\frac{\partial u}{\partial z} = \alpha LT, z = 0, \tau > 0 \quad \dots (11)$$

$$T(0, \tau) = T_o, \tau > 0 \quad \dots (12)$$

In addition to the above conditions we must have the regularity condition that the stress σ, displacement u and temperature T are bounded for all z and τ.

SOLUTION OF THE PROBLEM

Let us denote the Laplace transform with respect to τ, of any function $g(z,\tau)$ by $\overline{g}(z,p)$ where p is the transform parameter.

With this notation, Laplace transforms of (5) and (8), taking into account of initial conditions (9) and (10), yield

$$\frac{\partial^2 \overline{u}}{\partial z^2} - a\frac{\partial \overline{u}}{\partial z} - p^2\overline{u} = \alpha L\left(\frac{\partial \overline{T}}{\partial z} - a\overline{T}\right) \quad \dots (13)$$

and

$$\frac{\partial^2 \overline{T}}{dz^2} = \frac{p}{h}\overline{T} \quad \dots (14)$$

Laplace transform of boundary conditions (11) and (12) yield

$$\frac{\partial \overline{u}}{\partial z} = \alpha L\overline{T}, z = 0 \quad \dots (15)$$

$$\overline{T}(O, p) = \frac{T_0}{p} \quad \dots (16)$$

Solution of (14) with boundary condition (16) is given by Spiegel [7],

$$\overline{T}(z,p) = \frac{T_o}{p} e^{-\sqrt{\frac{p}{h}}z} \quad \dots (17)$$

In view of (17), differential equation (13) may now be solved for $\overline{u}$ as

$$\overline{u}\,(z,p\,) = A\,e^{\lambda z} - \frac{\alpha L T_o}{p}\,\frac{\left(\sqrt{\frac{p}{h}}+a\right)e^{-\sqrt{\frac{p}{h}}\,z}}{a\sqrt{\frac{p}{h}}+\frac{p}{h}-p^2} \qquad \ldots (18)$$

where the regularity condition $\overline{u}\,(\infty\,,p) = 0$ has been used. The quantity λ is given by

$$\lambda = \mu - \sqrt{p^2+\mu^2} \qquad \ldots (19)$$

where $\mu = a/2$.

The constant A may be determined by using (15) and (16). Finally, we obtain $\overline{\sigma}(z,p)$ as

$$\overline{\sigma}(z,p) = \alpha\,E_o E\,e^{-az}\left[e^{-z\sqrt{p/h}} - e^{\lambda z}\right]\frac{p}{\frac{p}{h}+a\sqrt{\frac{p}{h}}-p^2} \qquad \ldots (20)$$

If β_1, β_2, β_3, are the roots of the cubic

$$y^3 - b^2 y - a b^2 = 0 \qquad \ldots (21)$$

where $b = 1/h$ and $y = \sqrt{p/h}$, then (20) may be written as

$$\overline{\sigma}(z,p\,) = \alpha\,E_o T_o\,e^{-az}\left[e^{\lambda z}\sum_{i=1}^{3}\frac{L_i}{\sqrt{p}\left(\sqrt{p}+\beta_i\,\sqrt{h}\right)}\right.$$

$$\left. - \frac{e^{-z\sqrt{p/h}}}{h}\sum_{i=1}^{3}\frac{M_i}{p\left(\sqrt{p}-\beta_i\,\sqrt{h}\right)}\frac{1}{p}\,e^{-z\sqrt{p/h}}\right] \qquad \ldots (22)$$

where

$$L_i = \frac{\beta_i^2}{\left(\beta_j-\beta_i\right)\left(\beta_k-\beta_i\right)}$$

$$M_i = \frac{a-\beta_i}{(\beta_j-\beta_i)(\beta_k-\beta_i)},\ i \neq j \neq k,\ i, j, k = 1, 2, 3$$

Equation (22) may now be used to find the stress $\sigma(z, \tau)$ at any point z at time τ by applying inverse Laplace transform [cf. Erdelyi [2]]

$$\sigma(z,\tau) = \alpha E_o T_o e^{-az/2}\left[G(\tau-z)H(\tau-z) + \int_z^{\tau} f(u,z)\, G(\tau-u)\, du \right]$$

$$+ \alpha E_o T_o\, e^{-az}\left[g(\tau,z) - \text{erfc}\left(z/2\sqrt{h\tau}\right)\right] \qquad \dots (23)$$

where

$$f(\tau,z) = -\frac{\mu z}{\sqrt{\tau^2-z^2}} J_1\left(\mu\sqrt{\tau^2-z^2}\right) H(\tau-z)$$

$$G(\tau) = \sum_{i=1}^{3} L_i\, e^{\beta_i^2 h\tau}\, \text{erfc}\left(\beta_i\sqrt{h\tau}\right)$$

$$g(\tau,z) = \sum_{i=1}^{3} \frac{M_i}{h^{3/2}\beta_i}\left[\text{erfc}\left(z/2\sqrt{h\tau}\right) - e^{\beta_i z + \beta_i^2 h\tau}\, \text{erfc}\left(\frac{z}{2\sqrt{h\tau}} + \beta_i\sqrt{h\tau}\right)\right]$$

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2} dt$$

and $H(\tau)$ is Heaviside's step function.

NUMERICAL RESULTS

To get an idea about the effect of nonhomogeneity as stipulated in (3) on the stress, variations of stress with position and time as given by (23) has been numerically computed. Although the roots β_1, β_2, β_3 of the cubic in (21) may be found by any standard method, the restriction $b = 3\sqrt{3a}$ makes the procedure quite simple. In our numerical evaluation we have

used this condition and get $\beta_1 = -5.\ 63816\ a, \beta_2 = \ 4.59625\ a, \ \beta_3 =$ 1.04190 a.

In figures 1 and 2 we have shown the variations of

$$p = \frac{\sigma}{\alpha E_o T_o}$$

With τ over the range 0 to 6.0 for τ. In Fig. 1 the parameter a in (3) has been fixed at 0.2 and the variations of stress with positions have been shown. It is observed from the graph that at a particular position stresses diminish with time while within the range of τ considered the magnitude of stress at a particular time for points on the right of the rod is greater than that for the points in the left. From the nature of the graph it is apparent that for sufficiently large values of z he stress becomes negligible, which is in conformity with the regularity condition. In Fig. 2 we have plotted the variations of P with time keeping z fixed at 2. It is observed that with the increase of a the stress diminishes, while the behaviour of the stress remaining the same.

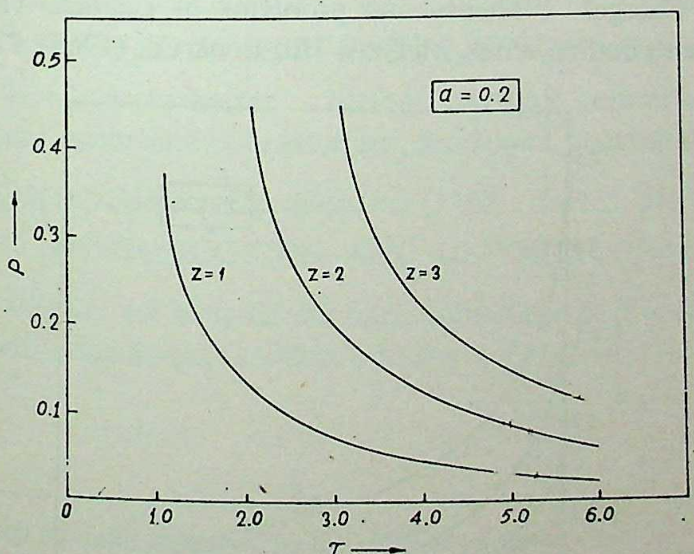

Fig. 1 Variation of stress with time (for fixed a)

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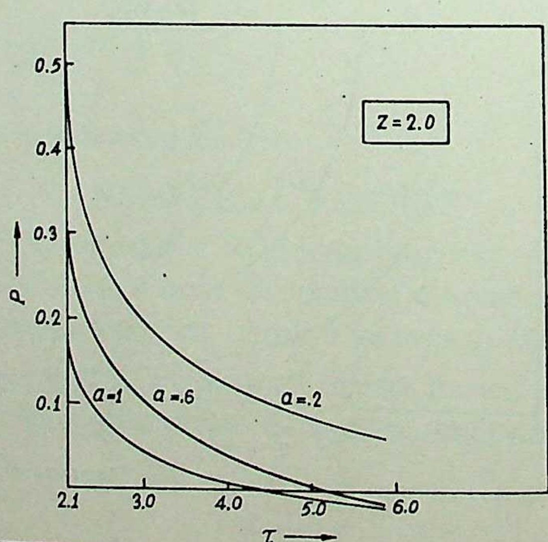

Fig. 2 Variation of stress with time (for fixed Z)

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COINCIDENCES AND FIXED POINTS OF MEIR-KEELER TYPE CONTRACTIVE MAPPINGS ON MENGER SPACES

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ABSTRACT

In this paper the notion of Meir-Keeler type contractive triplet of mappings on a Menger space is introduced and some coincidence and fixed point theorems for such mappings are established. The present results generalize a number of well-known results in fixed point theory.

Key words and phrases : Meir-Keeler type contractive triplet, Menger space, probabilistic metric space, fixed point, coincidence point.

AMS (MOS) Subject Classifications (1980) : 54H25, 54E99

INTRODUCTION AND DEFINITIONS

The well-known Meir-Keeler fixed point theorem [8] proved under the contractive condition (1) with A = M and T = I (Identity mapping) has

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been applied, extended and studied, among others by Chung [2] Ćirić [3] Maiti-pal [6], Matkowski-Wegrzyk [7], Park [9] Park-Bae [10], Park-Moon [11], Park-Rhoades [12], Rao-Rao [13], Rhoades [14], Shisheng [16], Singh [17], Singh-Virendra [20] and Yen-Chung [21]. For fixed point theorems for three mappings on a metric space satisfying Meir-Keeler type contractive conditions, refer to Park-Moon [11] and Rhoades [14]. However, fixed point theorems of these authors and several others, except of course [13] and [17], require continuity condition on the mapping(s) under consideration.

Recent coincidence theorems (on metric spaces) of Park [9] and Singh [17] extend and unify a coincidence theorem of Goebel [4], Meir-Keeler fixed point theorem [8] and a few fixed point theorems for contractive mappings.

In this paper we introduce Meir-Keeler type contractive conditions in Menger spaces and establish coincidence and fixed point theorems for three mappings, wherein either we use no continuity conditions or a very mild continuity condition is used.

In all that follows, A stands for an arbitrary nonempty set, (M,d) for a metric space, X for a Menger space (X, f, t) where t is continuous and satisfies $t(x, x) \geq x,\ x \in [0,1]$. By $C(P,T)$ we denote the set of coincidence points of mappings P and T, i.e., $C(P,T) = \{z \in A : Pz = Tz\}$.

Definition 1. A probabilistic metric space (PM-space) is an ordered pair (X, f) consisting of a nonempty set X and a mapping, f form $X \times X$ to L the collection of distribution functions. We shall represent the value of f at $(u,v) \in X \times X$ by $F(u, v)$. The functions $F(u, v)$ are assumed to satisfy the following conditions :

(a) $F(u,v; x) \quad = \quad 1$ for all $x > 0$, iff $u = v$;

(b) $F(u,v; 0) \quad = \quad 0$ for all $u,v \in X$;

(c) $F(u,v) \quad = \quad F(v,u)$ for all $u, v \in x$;

(d) if $F(u, v; x) = 1$ and $F(v, w; y) = 1$ then

$F(u, w; x+y) = 1.$

A Menger space is a triplet (X, f, t) where (X, f) is a PM-space and t-norm (or T-norm [15]) t is such that (d) is replaced by

(d') $F(u,w; x+y) \geq t\,(F(u,v; x), F\,(v,w,y))$

for all $x, y \geq 0$. For the details of topological preliminaries, refer to schweizer and Sklar [15].

Let P,Q,T be mappings on A with values in M or X as per situation. Consider the following conditions :

(1) For a given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ Such that for $u, v \in A$, $\varepsilon \leq d\,(Tu, Tv) < \varepsilon + \delta +$ implies $d\,(Pu, Pv) < \varepsilon$.

(2) For a given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for u, v $\in$ A,

$\varepsilon \leq \max\{d\,(Tu, Tv), d\,(Tu, Pu), d\,(Tv, Qv), 1/2\,[d(Tu, Qv)+d\,(Tv, Pu)]\}$

$< \varepsilon + \delta$ implies $d(Pu, Qv) < \varepsilon$.

The condition (1) has been used to generalize Goebel's coincidence theorm [4] by Park [9], while Park-Bae [10], using the condition (1) with A = M and PT = TP, have unified the fixed point theorems of Jungck [5] and Meir-Keeler [8]. A recent fixed point theorem of Park-Moon [11, Th.3.1], proved under the condition (2) with A = M, T continuous and commuting with each of P and Q extends and unifies a number of Meir-Keeler type fixed point theorems. Park-Moon [11] have also given a nice comparison of Meir-Keeler type contractive mappings. Note that (2) includes (1). Now, akin to (2) we introduce a Meir-Keeler type contractive condition for three mappings in a Menger space. Our results proved under this condition extend and include fully a number of results, among others, from [3-6], [8-10], [18], [19], and partially from [11-13], [17] and [20].

Let min $\{F(Tu, Tv; x), F(Tu, Pu; x), F\,(Tv, Qv; x)$

$F(Tu, Qv; 2x), F(Tv, Pu; 2x)\} = K(u, v; x)$ *for* u,v *in* A *and* $x > 0$.

Definition 2. Three selfmappings P,Q,T on X will be called a Meir-Keeler type contractive triplet if given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

(3) $0 = K(u, v; \varepsilon) < K(u,v; \varepsilon + \delta) = 1$ implies

$F(Pu, Qv; \varepsilon) = 1.$

Definition 3. Let P,Q,T be selfmappings on X If there exists a point u_0 in X and a sequence $\{u_n\}$ in X such that

$$Tu_{2n+1} = Pu_{2n}, Tu_{2n+2} = Qu_{2n+1}, n = 0,1,2,.....$$

then the space X is called $(P, Q; T(u_0))$ - orbitally complete with respect to μ_0 or simply $(P,Q; T)$ - orbitally complete if the closue of $\{T_{u_n} : n = 1,2,.......\}$ is complete.

Definition 4. The mapping T is called $(P,Q,; T(u_0))$ - orbitally continuous if the restriction of T on the closure of $\{Tu_n : n = 1,2,......\}$ is continuous.

Definition 5. Two selfmapings P and T on X will be called weakly commuting if,

$F(PTu, TPu; x) \geq F(Tu, Pu; x)$ *for all* $u \in X$ *and* $x > 0$.

RESULTS

The following lemma is motivated by S.– S. Chang [1].

Lemma. Let $\{y_n\}$ be a sequence in X such that $\lim_n F(y_n, y_{n+1}; \varepsilon) = 1$ for any $\varepsilon > 0$. If $\{Y_n\}$ is not Cauchy then there exist $\varepsilon_0 > 0$ and two sequences of positive integers, $\{m_i\}$ and $\{n_i\}$, such that

(a) $m_i > n_i + 1, i \to \infty, n_i \to \infty;$

(b) $F(y_{m_i}, y_{n_i}; \varepsilon_0) \leq 1 - \lambda, \quad 1 = 1, 2, ...,$

(c) $F(y_{m_i-1}, y_{n_i}; \varepsilon_0) > 1-\lambda, \quad i = 1, 2, ...$

Proof. Since $\lim_n F(y_n, y_{n+1}; \varepsilon) = 1$, for any $\varepsilon_0, \lambda > 0$, there exists a positive integer N such that $F(y_n, y_{n+1}; \varepsilon_0) > 1-\lambda$ for $n \geq N$. As the sequence is not cauchy, we have for $\varepsilon_0, \lambda > 0$,

$F(y_{n_1}, y_{p_1}; \varepsilon_0) \leq 1-\lambda, p_1 > n_1 > N_1 > N.$

Let $S = \{p \in I^+ : F(Y_{n_1}, Y_p; \varepsilon_0) \leq 1-\lambda, \ p > n_1\}$, I^+ being the set of positive integers. Then S is clearly nonempty. Let $p_0 \in S$ such that $p_0 \leq n_1 + 1$. Then $n_1 + 1 \geq p_0 > n_1 > N$, *i.e.*, $p_0 = n_1 + 1$. Thus,

$F(y_{n_1}, y_{p_0}; \varepsilon_0) = F(y_{n_1}, y_{n_1+1}; \varepsilon_0) > 1-\lambda$, a contradiction, implying $p > n_1+1, p \in S$. Now we can find a minimal integer $m_1 \in A$ such that $m_1 < n_1 + 1$, $F(y_{n_i}, y_{m_i}; \varepsilon_0) \leq 1 - \lambda$ and $F(y_{n_i}, y_{m_i}; \varepsilon_0) < 1-\lambda$. Repeated use of these arguments gives the desired conclusions.

Theorem 1. Let P, Q, T be a Meir-Keeler type contractive triplet on A with values in X. If there exists a point u_0 in A such that X is $(P, Q; T(u_0))$ – orbitally complete then P, Q, T have a coincidence, i.e., there exists a point z in A such that $Pz = Qz = Tz$. Further, if $A = X$ and for all $u, v \in X$, $x > 0$, either

(i) $F(Qu, Tv; x) \geq \min\{F(v, Qu; x), F(v, Tu; x)\}$

or

(ii) $F(Pu, Tv; x) \geq \min\{F(v, Tu; x), F(v, Pu; x)\}$ then

P, Q, T have a unique common fixed point and $\{Tu_n\}$ converges to the fixed point.

Proof. Condition (3) is equivalent to

(4) $F(Pu, Qv; x) > \min\{F(Tu, Tv; x), F(Tu, Pu; x), F(Tv, Qv; x), F(Tu, Qv; 2x), F(Tv, Pu; 2x)\}$.

Two cases arise.

Case I. suppose $Tu_n = Tu_{n+1}$ and $Tu_{n+1} \neq Tu_{n+2}$ for some odd integer n. By (4)

$$F(Tu_{n+2}, Tu_{n+1}; x)$$

$$= F(Pu_{n+1}, Qu_n; x)$$

$$> \min \{F(Tu_{n+1}, Tu_n; x), F(Tu_{n+1}, Tu_{n+2}; x), F(Tu_n, Tu_{n+1}; x), F(Tu_{n+1}, Tu_{n+1}; 2x), F(Tu_n, Tu_{n+2}; 2x)\}$$

$= F(Tu_{n+1}, Tu_{n+2}; x)$, since by (d')

$$F(Tu_n, Tu_{n+2}; 2x) \geq \min \{F(Tu_n, Tu_{n+1}; x), F(Tu_{n+1}, Tu_{n+2}; x)\}.$$

But this is a contradiction.

Hence $Tu_{n+1} = Tu_{n+2}$, *i.e.*, in general

$$Tu_{n+i} = Tu_n,\ i = 1, 2, \ldots$$

Taking $u_n = z$, we have

(5) $Tu_n = Tu_{n+1} = Qu_n$, *i.e.*, $Tz = Qz$.

Let $Pz \neq Qz$, then from (4),

$F(Pz, Qz; x) > F(Pz, Qz; x)$, a contradiction.

Thus, $Pz = Qz$. Hence, from (5),

$$Pz = Qz = Tz.$$

The case when n is even may be dealt with similarily.

Case II. Suppose $Tu_n \neq Tu_{n+1}$ for all integers n. The application of (4) gives that $\{F(Tu_n, Tu_{n+1}; x)$ is an increasing sequence converging to a number $0 \leq r \leq 1$. Let $r < 1$.

From this we infer that for every $s \in (0, r)$ there exists a positive integer N such that $r - s \leq F(Tu_n, Tu_{n+1}; x) < r$ for all $n \geq N$, since the convergence is monotone from the left.

Let $2k \geq N$, then by (4)

$$F(Tu_{2k+1}, Tu_{2k+2}; x) > \min\{F(Tu_{2k}, Tu_{2k+1}; x), F(Tu_{2k}, Pu_{2k}; x),$$
$$F(Tu_{2k+1}, Qu_{2k+1}; x), F(Tu_{2k}, Qu_{2k+1}; 2x),$$
$$F(Tu_{2k+1}, Pu_{2k}; 2x)\} = F(Tu_{2k}, Tu_{2k+1}; x).$$

Thus $F(Tu_{2k+1}, Tu_{2k}+2; x) > r$, a contradiction implying $r = 1$.

To prove that the sequence $\{Tu_n\}$ is a Cauchy sequence, suppose the contrary to be true. Then, by the Lemma, there exists $\varepsilon_0 > 0$ and two sequences of positive integers,

$\{m_i\}$ and $\{n_i\}$, such that

(a) $m_i > n_i + 1, n_i \to \infty, i \to \infty$;

(b) $F(Tu_{m_i}, Tu_{n_i}; \varepsilon_0/2) \leq 1 - \lambda, i = 1, 2, \ldots$;

(c) $F(Tu_{m_i-1}, Tu_{n_i}; \varepsilon_0/2) > 1 - \lambda, i = 1, 2, \ldots$.

Thus using (b), (c) and passing on to the limits we have

$$F(Tu_{m_i}, Tu_{n_i}; \varepsilon_0) \geq \min\{F(Tu_{m_i}, Tu_{m_i-1}; \varepsilon_0/2),$$
$$F(Tu_{m_i-1}, Tu_{n_i}; \varepsilon_0/2)\} > \min\{1, 1 - \lambda\} = 1 - \lambda,$$

a contradiction.

Therefore $\{Tu_n\}$ is a Cauchy sequence. Since $\times$ is $(P, Q, T(u_0))$ – orbitally complete, there exists a z in $\times$ such that $Tu_n \to z$.

Now if (i) holds, then

$F(Qu_{2n+1}, Tz; x) \geq \min\{F(z, Qu_{2n+1}; x), F(z, Tu_{2n+1}; x)\}$, implying $Tz = z$.

And if (ii) holds then also $Tz = z$.

Also, since $Tu_m \neq Tu_{m+1}$, we get by (4),

$F(z, Qz; x) > F(z, Qz; x)$, giving $z = Qz$ which in trun, again with the application of (4), gives $z = Pz$.

Thus $\quad Pz = Qz = Tz = z.$

With suitable choice of P, Q, T in (3) we get probabilistic analogues of various well-known coincidence and fixed point theorems in metric spaces, some of them being due to Goebel [4], Meir-Keeler [8], Park [9], Park-Bae [10], Rao-Rao [13], Singh [17] and Singh-Virendra [20].

***Theorem* 2.** Let $P, Q, T : X \to X$ be a Meir - Keeler type contractive triplet. If there exists a point u_0 in X such that X is $(P, Q; T(u_0))$ – orbitally complete and,

either (i) $PTz = TPz, z \in C(P, T)$;

or (ii) $QTz = TQz, z \in C(Q, T)$, then P,Q,T, have a unique common fixed point and $\{Tu_n\}$ converges to the fixed point.

Proof. The uniquene of the fixed point of P, Q, T if it exists, is ensured by (4) with $A = X$. So, in view of Case I of the proof of Theorem 1, it is sufficient to show that P, Q, T have a common fixed point $Tz(=Pz=Qz)$. Without loss of generality we may assume that P and T commute at $z \in C(P, T)$.

Suppose (i) holds. Then

(6) $\quad PTz = TPz, = TTz, \text{ as } Tz = Pz.$

Now taking $TTz \neq Tz$, we have by (4),

$$F(TTz, Tz; x) = F(PTz, Qz; x) > F(TTz, Tz; x), \text{ a contradiction.}$$

Thus, in view of (6)

$$TTz = Tz = PTz.$$

Clearly $QTz = Tz$, since otherwise it leads to a contradiction. Hence $Tz(=Pz=Qz)$ is a common fixed point of P, Q, T. Hence Tz $(=Pz=Qz)$ is a common fixed point of P, Q, T.

Remark: It is interesting to note that every pair of mappings (P, T) commuting weakly on $C(P,T)$ is commuting on $C(P,T)$. Thus weak commutativity of P and T at a coincidence point z of P and T is equivalent to the commutativity of P and T at z. Therefore Theorem 2 holds even if commutativity of P and T (resp. Q and T) is replaced by weak commutativity of P and T (resp. Q and T) Thus the proof of the following theorem is obvious.

Theorem 3 . Let $P, Q, T, : X \to X$ be a Meir-Keeler type contractive triplet. If there exists a point u_0 in X such that X is $(P, Q,; T(u_0))$ – orbitally complete and either

(i) P and T are weakly commuting and $(P,Q;T(u_0)$ –orbitally continuous;

or

(ii) Q and T are weakly commuting and $(P, Q; T(u_0))$ –orbitally continuous, then P, Q, T have a unique common fixed point p (say) in X and for some $u_0 \in X$, the sequence $\{Tu_n\}$ converges to a point z in X such that $Tz = p$.

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LOGISTIC GROWTH UNDER RANDOM DRIVING FORCES

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ABSTRACT

The stochastic behaviour of non-equilibrium fluctuation in the logistic model of growth under random driving forces has been studied in the light of the 'fluctuation-dissipation' theorem of statistical mechanics.

Key words : Logistic growth equation, Carrying capacity, Non-equilibrium fluctuation, White-noise, Fluctuation-dissipation theorem.

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1. INTRODUCTION

Logistic growth equation plays an unique place as the first non-linear equation for population growth. It finds wide application in different branches of biology. It is also the basic equation of non-linear autocatalytic reactions which is fundamental in understanding many phenomena of life processes [7]. A lot of works was done on this equation covering various aspect of population, but there are others to be studied from a different consideration.

In this paper we shall study the stochastic behaviour of non-equilibrium fluctuation in the logistic growth of a biological system under

the influence of random driving force. The randomness appeared here is assumed to be internal, that is, due to the random interactions among the individuals of a single species [3]. Two cases namely, the state with constant carrying capacity and the other with time-varying capacity have been considered.

2. VERHULST GROWTH UNDER RANDOM DRIVING FORCE

A. Constant Carrying Capacity

The Verhulst (or Logistic) equation of growth of a single species population is given by

$$\frac{dN}{dt} = \alpha N - \beta N^2 = \alpha N \left(1 - \frac{N}{K}\right) \qquad \ldots (1)$$

where $N(t)$ is the population size at any time t, α is the intrinsic rate constant and $K = \alpha/\beta$ is the carrying capacity of the environment. Let us first assume that K or β is constant. This type of equation occurs in different cases of biophysical system [7]. The stochastic extension of the equation (1) is given by

$$\frac{dN}{dt} = \alpha N - \beta N^2 + \varepsilon(t) \qquad \ldots (2)$$

where $\varepsilon(t)$ is the random perturbation term which is due to the overall effect of the numerous microscopic, unknown or partially known internal fluctuation of the state variable N. We have called the randomness of perturbation to be internal in the sense that it is due to the internal effect of mutual interactions of the individuals of the same species without any consideration of the randomness of the environmental parameters. The random perturbation term $\varepsilon(t)$ is assumed to be a white noise satisfying the condition

$$\langle \varepsilon(t) \rangle = 0, \quad \langle \varepsilon(t_1)\, \varepsilon(t_2) \rangle = 2D_0\, \delta(t_1 - t_2) \qquad \ldots (3)$$

where < > represents the average over the ensemble of the stochastic process. D_0 is the diffusion coefficient. The equation (2) is a non-linear stochastic differential equation of non-linear Langevin type and there are some difficulties in solving such problem [6]. We shall, however, resort to the linearization method to study the characteristic stochastic behaviour of the system. The equation (2) is linearized as follows :

The equation for the average $\langle N \rangle$ is given by

$$\frac{d\langle N\rangle}{dt} = \alpha \langle N\rangle - \gamma \langle N^2\rangle \qquad \text{...(4)}$$

or, approximately by

$$\frac{d\langle N\rangle}{dt} = \alpha \langle N\rangle - \gamma \langle N\rangle^2 \qquad \text{... (5)}$$

Where we have used the relation $\langle N^2\rangle = \langle N\rangle^2$ as a first approximation.

Let us put $N(t) = \langle N\rangle + N_0(t)$ ... (6)

in equation (2) and subtracting (4) from it, we get the linear equation

$$\frac{dN_0}{dt} = \left(\alpha - 2\beta\langle N\rangle\right)N_0 + \varepsilon(t) \qquad \text{... (7)}$$

Where we have neglected second order terms of $N_0(t)$.

If $\sigma^2(t) = \left\langle (N - \langle N\rangle)^2\right\rangle = \left\langle N_0^2\right\rangle$

the $\sigma^2(t)$ satisfies the ordinary differential equation [2]

$$\frac{d}{dt}\sigma^2(t) = 2\left(\alpha - \beta\langle N\rangle\right)\sigma^2(t) + 2D_0 \qquad \text{... (8)}$$

For $\alpha > 0$ and for large t, $\langle N\rangle \to 0$ so that the equation (8) takes the form

$$\frac{d\left\langle N_0^2(t)\right\rangle}{dt} = 2\alpha \left\langle N_0^2(t)\right\rangle + 2D_0 \qquad \text{... (9)}$$

$$\text{Then } \left\langle N_0^2(t)\right\rangle\Big|_{t \to \text{large}} = \frac{D_0}{\alpha} \qquad \text{... (10)}$$

B. Quasi-steady State : Fluctuation

Let us now consider the situation of the quasi-steady state which corresponds to the state very near to the steady non-equilibrium state. Such a state is characterized by the slow variation of the drift coefficient $A(t)\left(\alpha - \beta \langle N \rangle\right)$ and the diffusion coefficient D_0 which is assumed here to be constant. In addition, the addmittance functionn $Z(i\omega, t)$ relating $N_0(t)$ and $e(t)$ of the stochastic equation (7) must be slowly varying function of time. Under these assumptions, the mean square fluctuation of the population size $N(t)$ is given by [1]

$$\left\langle N_0^2(t)\right\rangle = \frac{D_0}{\pi}\int_{-\infty}^{\infty}\left|\frac{A(t)}{\left(A(t)+i\omega\right)} + \frac{A(t)}{\left(A(t)+i\omega\right)^3}\right|^2 d\omega \qquad \text{... (11)}$$

The above formula is a generalization of the famous Nyguist's relation for quasi-stationary state [1]. For the slowly varying condition of the drift coefficient $A(t) = \left(\alpha - \beta \langle N \rangle\right)$, We must have

$$\left|\frac{\dot{A}(t)}{A(t)}\right| << 1 \qquad \text{... (12)}$$

$$\text{or, } \left|\beta \langle N \rangle\right| << 1 \qquad \text{... (13)}$$

Under this condition let us now find the value of $\left\langle N_0^2(t)\right\rangle$. Now (11) can be reduced to the form

$$\left\langle N_0^2(t)\right\rangle = \frac{D_0}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\left(A^2(t)+\omega^2\right)} - \frac{2\dot{A}\cdot(t)}{\left(A^2(t)+\omega^2\right)^2}\right.$$

$$\left. + \frac{4\dot{A}(t)A^2(t)+\dot{A}^2(t)}{\left(A^2(t)+\omega^2\right)^3}\right]d\omega \qquad \ldots (14)$$

Since $A(t)=\alpha-\beta\left\langle N\right\rangle, \dot{A}(t)=-\beta\left\langle N\right\rangle A(t)$

So

$$\left\langle N_0^2(t)\right\rangle = \frac{D_0}{\pi}\left[\int_{-\infty}^{\infty}\frac{d\omega}{A^2(t)+\omega^2} + 2\beta A(t)\langle N\rangle\int_{-\infty}^{\infty}\frac{d\omega}{\left(A^2(t)+\omega^2\right)^2}\right.$$

$$\left. + \left\{\beta^2\langle N\rangle^2A^2 - 4\langle N\rangle A^3\beta\right\}\int_{-\infty}^{\infty}\frac{d\omega}{\left(A^2(t)+\omega^2\right)^3}\right] \qquad \ldots (15)$$

Evaluating the integrals in (15), we have

$$\left\langle N_0^2(t)\right\rangle \simeq D_0\left[\frac{1}{A(t)} - \frac{\beta\langle N\rangle}{2A^2(t)} + \frac{3\beta^2\langle N\rangle^2}{8A^3(t)}\right]$$

$$\simeq \frac{D_0}{\alpha}\left\langle 1+\frac{\beta\langle N\rangle}{\alpha}\right\rangle$$

$$= \frac{D_0}{a} + \frac{D_0}{a^2}\beta\langle N\rangle \qquad \ldots (16)$$

neglecting higher powers of $\beta\langle N\rangle$. Then for small $\beta\langle N\rangle$ under the condition (13), the result (16) for the quasisteady state agrees with the result (10) for large time t, as it should be.

3. VERHULST GROWTH UNDER RANDOM AND DETERMINISTIC DRIVING FORCES

A. *Periodic Carrying Capacity : Linearization*

Let us now consider the Verhulst equation (1) where the capacity K is now assumed to be time-dependent and this due to the variability of the environmental condition. Let us take $K(t)$ to be of sinusoidal form

$$K(t)=K_0(1+b\cos\omega t) \qquad \dots (17)$$

Then the equation (1) becomes

$$\frac{dN}{dt}=\alpha N\left[1-\frac{N}{K_0\left(1+b\cos\omega t\right)}\right] \qquad \dots (18)$$

The equation (18) can be scaled as follows

We write $N' = \frac{N}{N_0},\ t' = \alpha t\cdot$ ... (19)

Then the equation (18) reduces to the form

$$\frac{dN'}{dt'}=N'\left[\frac{N}{\left(1+b\cos\omega' t'\right)}\right] \qquad \dots (20)$$

where $\omega' = \frac{\omega}{\alpha}$ ... (21)

The steady state value for N' as seen from (20) is given by

$$N'+1=b\cos\omega' t' \qquad \dots (22)$$

For $b \ll 1$ and for long time interval the steady value of N' is one. Then writing $N' = 1+y$ and expanding about the steady value, the linear equation that results from (20) is given by [5]

$$\frac{dy}{d\,t'} + y = b\cos\ \omega\,'\,t\,' \qquad \text{... (23)}$$

The equation (23) is the basic deterministic equation and is a linear equation in y with a driving time-dependent force $b \cos \omega' t'$.

B. Stochastic Differential Equattion : Fluctuation

The stochastic extension of (23) is obtained by addition of random perrturbation term $\varepsilon(t')$ to the right hand side of (23).

$$\frac{d\,t}{d\,t'} + y = b\,\omega\,'\,t\,' + \varepsilon(t') \qquad \text{... (24)}$$

Where random noise $\varepsilon(t')$ is assumed to be a gaussian white noise satisfying the conditions

$$\left\langle \varepsilon\left(t_1'\right)\right\rangle = 0,\ \left\langle \varepsilon\left(t_1'\right)\varepsilon\left(t_1'\right)\right\rangle = 2D\,\delta\left(t_1' - t_2'\right) \qquad \text{... (25)}$$

Where < > represents the average over the ensemble of the stochastic process. The equation (24) is an extension of Langevin equation driven by a periodic force $b \cos \omega' t'$ and a random force $\varepsilon(t')$. The stochastic behaviour of the system satisfying the equation (24) can be studied from the extended form of the famous 'fluctuation- dissipation theorem' of non-equilibrium statistical mechanics [4].

To find the correlation function for the system satisfying the stochastic equation (24) let us assume that the system has evolved from the state $x = 0$ at some very distance time past $t = -\infty$ under the action of the time-varying external force $b \cos \omega' t'$. Then following Lavenda [4] the auto - correlation function is given by

$$C\left(t', t_0'\right) = \left\langle y\left(t'\right) y\left(t_0'\right)\right\rangle$$

$$= D \exp\left\{-\left(t' - t_0'\right)\right\} = \int_{-\infty}^{t'} \exp\left\{-\left(t' - u\right)\right\} b\cos\ \omega' u\ du$$

$$\int_{-\infty}^{t_0'} exp\left\{-\left(t_0'-u\right)\right\} b\cos\ \omega' u\ du \quad \text{... (26)}$$

$$= D\,exp\ \left(t_0'-t'\right) + b\,\frac{\left(\omega'\sin\ \omega' t' + \cos\ \omega' t'\right)}{1=\omega'}$$

$$b\,\frac{\left(\omega'\sin\ \omega' t_0' + \cos\ \omega' t_0'\right)}{1+\omega'} \quad \text{... (27)}$$

From (27) the average fluctuation $\left\langle y^2\left(t'\right)\right\rangle$ is given by

$$\left\langle y^2\left(t'\right)\right\rangle = D + \frac{b^2}{\left(1+\omega'\right)^2}\left(\omega'\sin\ \omega' t' + \cos\ \omega' t'\right)^2 \quad \text{... (28)}$$

which consists of two parts. The first part is due to the random force and the second part is due to the driving force, i.e., due to the environmental variability coming through the time-dependent carrying capacity. The average of $y\ (t')$ is given by

$$\left\langle y\left(t'\right)\right\rangle = \int_{-\infty}^{t'} exp\left\{-\left(t'-u\right)\right\} b\cos\left(\omega' u\right) du$$

$$= \frac{b}{1+\omega'}\left(\omega'\sin\omega' t' + \cos\omega' t'\right) \quad \text{... (29)}$$

which is the particular part of the solution of the deterministic equation (23), the transient part of the general solution vanishes for large value of the time t.

The mean square fluctuation of $y(t')$ is then given by

$$\sigma^2(t') = \langle y^2(t') \rangle - \langle y(t') \rangle^2 = D \qquad \text{... (30)}$$

The expression shows that the driving force $b \cos \omega' t'$ for the linearized equation (23) has no effect on the dispersion of the growth variable. The fluctuation arises as a result of the random force only. This, infact, follows as a consequence of the Gaussian-Markovinian assumption.

4. Conclusion

The paper aims to study the stochastic behaviour of non-equilibrium fluctuation in logistic non-linear growth of a biological system by the technique of linearization. Two problems have been studied - one is the fluctuation for constant carrying capacity and the other for the periodic carrying capacity of the environment. The paper is concerned with an important biological application of famous 'fluctuaion dissipation theorem' of statistical mechanics. In the first case it is based on a modified form of Nyguist's theorem for quasi - steady state. The result obtained in this case is a modification of the earlier well established and experimentally verified result of population fluctuation for constant carrying capacity. In the second case of periodic carrying capacity the theory is based on the extended form of "fluctuation-dissipation-theorem" for driving system. The results obtained in this case are in good agreement with those of Nisbet and Gurney [5] who justified their results in the light of Jillson's experiment on the culture of Tribolium Castaneum.

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AN EXTENSION OF ANCIENT INDIAN CUBE-ROOT METHOD

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ABSTRACT

In this paper an attempt is made to extend Bhāskarācārya's method of obtaining cube-root of a perfect cubic number to 5th root of any integer. The method indeed may be applied to compute n^{th} root of any positive integer.

1 : INTRODUCTION

Ancient Indian Mathematicians made an attempt to find out the cube-root of perfect numbers. For a detailed discussion refer to Līlāvatī of Bhāskara II (see also Bag [1], Datta and Singh [2], Jha [3], Mishra [7] and Naimpally *et al.* [8]. An attempt has been made by Lal [6] to give a chronological description of the methods for obtaining the real cube root of an integer given by the Hindu Mathematicians. Bhāskara II, in his Līlāvatī, gave a method to find out (real) cube-root of a positive integer. It seems that ancient Indian Pandits knew cube-root techniques much before Bhāskara (b.1114 A.D.). Bhāskara's method is slightly different from his predecessors.

In this paper we make an attempt to give a general method to compute real n^{th} root of an integer. Apart from including several examples to illustrate the technique of computing 5th, 6th, 7th roots of a number,

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'quick computation' is also indicated in certain cases, particularly when the root to be computed is of two digits.

Definition.: A number is said to be a perfect number of order n (or simply a perfect number in this paper) if it is expressible in the form of x^n, where n is a positive integer and x is any rational number.

First look at the following table. Calculations at various steps become a little easier if one remembers the message of this table.

TABLE -1

SHOWING END OF ROOTS OF PERFECT NUMBERS ACCORDING TO THEIR UNIT PLACES

Num- ber	Unit places in different powers														
	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	8	6	2	4	8	6	2	4	8	6	2	4	8
3	3	9	7	1	3	9	7	1	3	9	7	1	3	9	7
4	4	6	4	6	4	6	4	6	4	6	4	6	4	6	4
5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
7	7	9	3	1	7	9	3	1	7	9	3	1	7	9	3
8	8	4	2	6	8	4	2	6	8	4	2	6	8	4	2
9	9	1	9	1	9	1	9	1	9	1	9	1	9	1	9

Note that :

1. 5th, 9th, 13th, 17th etc. powers of 1, 2, 3, 4, 5, 6, 7, 8 & 9 end up respectively in 1, 2, 3, 4, 5, 6, 7, 8 & 9.
2. 6th, 10th, 14th, 18th etc. powers of 1, 2, 3, 4, 5, 6, 7, 8 & 9 end up respectively in 1, 4, 9, 6, 5, 6, 9, 4 & 1.

3. 7th, 11th, 15th, 19th etc. powers of 1, 2, 3, 4, 5, 6, 7, 8 & 9 end up respectively in 1, 8, 7, 4, 5, 6, 3, 2, & 9.

4. 8th, 12th, 16th, 20th etc. powers of 1, 2, 3, 4, 5, 6, 7, 8 & 9 end up respectively in 1, 6, 1, 6, 5, 6, 1, 6 & 1.

5TH ROOT OF A POSITIVE INTEGER

Note that fifth power of number 1 through 9 are respectively 1, 32, 243, 1024, 3125, 7776, 16807, 32768 and 59049

Given a perfect positive integer of order 5, to find out its fifth positive real root.

Step I : Put a dash on the unit place and a dot on each of the next four digits. Repeat this process till the last digit of the number. Note that the number of dashes equals the number of digits (before decimal) in the proposed fifth root. For example, if the given number consists of 13 digits, then it is divided by the "dash-dot" method into 3 blocks and the first two blocks consist of 5 digits each and the last block consists of only 3 digits.

Step II : From the last block subtract the 5^{th} power of the greatest possible positive integer, say a.

Step III : Associate with this remainder, the first digit of the next block. Now divide this new number by $5a^4$. Let the quotient be b. Evidently $0 \le b \le 9$ (see Example 2.1 wherein 4 is associated with the remainder 111 to give a new number 1114, and b = 7). This new number may be called 2nd Pada (or Pada II), while the (last) block with which we started the actual computation may be called "first Pada" (or simply Pada I).

Step IV : Form Pada III by taking down this new remainder and the next digit. (For example, our third Pada in Example 2.1 is 5548). Subtract $10a^3b^2$ from Pada III, $10a^2b^3$ from Pada IV, $5ab^4$ from Pada V and b^5 from Pada VI. (If subtraction of $10a^3b^2$ is not **possible, reduce b of Step III** until subtraction becomes possible, **and now the subtraction** is done with the reduced b).

Step V : If more blocks are left out divide Pada VII by $5(10a+b)^4$ to get quotient c (where $c \leq 9$). Subtract $10\,(10a+b)^3c^2$ from Pada VIII, $10\,(10a+b)^2c^3$ from Pada IX, $5(10a+b)c^4$ from Pada X and c^5 from Pada XI respectively. (If subtraction of $10\,(10a+b)^3c^2$ is not possible, c is reduced and adjusted in the same way as b).

Step VI : If more blocks are still left out, divide the next Pada XII by $5\,(100a+10b+c)^4$ to get quotient d where d (≤ 9) is adjusted, if necessary, in the same way as b and c). Subtract $10\,(100a+10b+c)^3d^2$, $10(100a+10b+c)^2d^3$, $5\,(100a+10b+c)d^4$ and d^5 from the next Padas respectively.

The process may be extended in this fashion if more blocks are still left out.

Results : (i) For the number having two blocks root is 10a+b, (where b is at the unit place for perfect numbers only).

(ii) For a number having three blocks (respectively four blocks), root is 100a+10b+c (respectively 1000a+100b+ 10c+d) wherein c (respectively d) is at the unit place.

Now we illustrate the method by examples.

Example *2.1* Find the 5[th] real root of 14348907.

Computation may be done as follows wherein a and b assumed in Steps II-V are given in this and subsequent examples only for the sake of clarity.

1 4 3 4 8 9 0 7

$a^5 = 2^5$ $\qquad$ $2 = a$

32

$5a^4 = 5.2^4 = 80$) 1114 ($7 = b$

$5a^4b = 560$ $\qquad$ 560

5548

$10a^3b^2 = 10.2^3.7^2$

$= 3920$ $\qquad$ 3920

16289

$10a^2b^3 = 10.2^2.7^3$

$= 13720$ $\qquad$ 13720

25690

$5ab^4 = 5.2.7^4$
$= 24010$ $\quad\quad$ $\underline{24010}$
$\quad\quad\quad\quad 16807$

$b^5 = 7^5$
$= 16807$ $\quad\quad$ $\underline{16807}$
$\quad\quad\quad\quad \times$

Hence 5^{th} root of 14348907 is 27.

It is important to note that, in view of the observations made on Table 1,7 seems to be at the unit place of the proposd 5^{th} root. Since there are two blocks, the answer will comprise of only two digits. Consequently the Calculations beyond Pada II are not needed, if we know that the given number is a perfect number of 5^{th} order.

Example 2.2 Find 5^{th} root of 4747561509943.

We have the following unnarrated steps showing the details of calculations.

$$\dot{4}\dot{7}\ 4\ 7\dot{5}\dot{6}\dot{1}\ \dot{5}\ '\ \dot{0}\dot{9}\dot{9}\dot{4}\ \dot{3}$$

$a^5 = 3^5$ $\quad\quad$ $3 = a$

$\quad\quad 243$

$5a^4 = 5.3^4 =$ $\quad 405\overline{)\,2317\,}(\ 4 = b$

$10\,a^3b^2 = 10.3^2.4^2$ $\quad \underline{1620}$
$= 4320$ $\quad 6975$
$\quad \underline{4320}$
$10\,a^2b^3 = 10.3^2.4^3$ $\quad 26556$
$= 5760$ $\quad \underline{5760}$
$5ab^4 = 5.3.4^4$ $\quad 207961$
$= 3840$ $\quad \underline{3840}$
$\quad 2041215$
$b^5 = 4^5 = 1024$ $\quad 1024$

$5(\,10a + b)^4 = 5\,(34)^4 = 6681680$ $\quad \overline{)\,20401910\,}(\ 3 = c$
$10(10a+b)^3\,c^2 = 10(34)^3\,.3^2$ $\quad \underline{20045040}$
$= 3537360$ $\quad 3568709$
$\quad \underline{3537360}$
$10\,(10s+b)^2\,c^3 = 10(34)^2\,.3^3$ $\quad 313499$
$= 312120$ $\quad \underline{312120}$
$5(10a+b)\,c^4 = 5(34)\,.\,3^4$ $\quad 13794$
$= 13770$ $\quad \underline{13770}$
$\quad 2'43$

$c^5 = 3^5 = 243$ $\qquad\qquad \underline{243}$

$\times$

Hence 5^{th} root of 4747561509943 is 343.

Example 2.3 : Find 5^{th} root of 14693280768.

$$
\begin{array}{llll}
 & & 1\ 4\ 6\ 9\ 3\ 2\ \ 8\ 0\ 7\ 6\ 8 & 1 = a \\
a^5 = 1^5 & & & \\
 & & 1 & \\
 & 5a^4 = 5.1^4 = & 5)\,46 & (08 = b \\
10a^3 b^2 & = 10.1^3 . (08)^2 & \underline{40} & \\
 & & 693 & \\
 & = 640 & \underline{640} & \\
10a^2 b^3 & = 10\,(1)^2\,(08)^3 & 5328 & \\
 & = 5120 & \underline{5120} & \\
5ab^4 & = 5.1.\,(08)^4 & 20807 & \\
 & = 20480 & \underline{20480} & \\
b^5 & = (08)^5 & 32768 & \\
 & = 32768 & \underline{32768} & \\
 & & \times &
\end{array}
$$

Here 4 is not divisible by 5 and so we put 0 as quotiant and bring down the next digit 6. Note that 46 is our Pada II.

Hence 5th root of 14693280768 is 108.

3 : 5^{th} ROOT METHOD FOR NON-PERFECT NUMBERS

Step I : Put dashes and dots as mentioned earlier to subdivide the number into blocks of five.

Step II : Put decimal and zeros after the unit place.

Example 3.1 : Find 5^{th} root of 2.

We do it as before.

$a^5 = 1^5$ 2.00000

$5a^4 = 5.1^4 =$ $5)\overline{10}($ $1 = b$ $1 = a$

$10a^3 b^2 = 10.(1)^3.(1)^2 = 10$

$10a^2 b^3 = 10.1^2.1^3 = 10$

$5ab^2 = 5.1.1^4 = 5$

$b^5 = 1^5 = 1$

$5(10a + b)^4 = 5.(11)^4 = 73205$

	1
5)	10 (1=b
	5
	50
	10
	400
	10
	3900
	5
	38950
	1
)	389490 (5=c
	366025
	23465 (Remainder)

Hence 5^{th} root of 2 (till two places of decimal) is 1.15.

Example 3.2 : Find 5th root of 68949

We follow again the previous method.

$a^5 = 9^5$ 68949 .000000 $9 = a$

		68949 .000000
$a^5 = 9^5$		59049
$5a^4 = 5.9^4 =$	32805	) 99000 (2=b
		65610
		333900
$10a^3 b^2 = 10.9^3.2^2 = 29160$		29160
		3047400
$10a^2b^3 = 10.9^2.2^3 = 6480$		6480
		30409200
$5ab^4 = 5.9.2^4 = 720$		720
		304084800
$b^5 = 2^5 = 32$		32

$$5\,(10a + b)^4 = 5(92)^4 = \quad 177065680 \;\big)\; 3040847680 \;(\; 9 = c$$
$$\underline{1593591120}$$
$$14472565600 \quad \text{(Remainder)}$$

We were looking for 9.29 (correct to two places of decimal).

Example 3.3 : Find 5th root of 34867. 84401.
(Given that this is a perfect number of order 5.)

$$\dot{3}\dot{4}8\dot{6}7\,.\,8\;4\;4\;0\;1$$

$$\begin{array}{lrl}
a^5 = 8^5 & 32768 & 8 = a \\
5a^4 = 5.8^4 = & 20480\big)\;20998\;(\;1 = b & \\
 & \underline{20480} & \\
10a^3\,b^2 = 10.\,8^3\,.\,1^2 & 5184 & \\
\quad = 5120 & \underline{5120} & \\
10a^2\,b^3 = 10.\,8^2\,.\,1^3 & 644 & \\
\quad = 640 & \underline{640} & \\
5ab^4 = 5.\,8\,.\,1^4 & 40 & \\
\quad = 40 & \underline{40} & \\
 & 1 & \\
b^5 = 1^5 & \underline{1} & \\
 & \times &
\end{array}$$

Hence 5^{th} root of 34867. 84401 is 8.1

4 : 6TH ROOT

Example 4.1 : Find 6^{th} root of 191102976.

$$1\dot{9}\;\dot{1}\;\;\dot{1}0\dot{2}9\dot{7}\;6$$

$$\begin{array}{lr}
a^6 = 2^6 & 64 \\
6a^5 = 6.2^5 = & 192\big)\;1271\;(\;4 = b \\
 & \underline{768} \\
15a^4\,b^2 = 15.2^4.4^2 & 5030 \\
\quad = 3840 & \underline{3840} \\
 & 11902 \\
20a^3b^3 = 20.2^3.4^3 = 10240 & \underline{10240} \\
 & 16629 \\
15a^2\,b^4 = 15.2^2.4^4 = 15360 & \underline{15360} \\
 & 12697
\end{array}$$

$6a.b^5 = 6.2.4^5 = 12288$

$b^6 = 4^6 = 4096$

```
 12288
  4096
  4096
     ×
```

Hence 6^{th} root of 191102976 is 24.

Example 4.2 : Find 6^{th} root of 17685.

$a^6 = 5^6$; $5 = a$

$6a^5 = 6.5^5 = 18750$

$15a^4b^2 = 1^5.5^4.1^2 = 9375$

$20\,a^3b^3 = 20.5^2.1^3 = 2500$

$15a^2b^4 = 15.5^2.1^4 = 375$

$6\,a.b^5 = 6.5.1^5 = 30$

$b^6 = 1^6 = 1$

$6\,(10a + b)^5 = 6\,(51)^5 =$

```
                 17685.000000
                 15625                      5 = a
           18750) 20600 (1=b
                  18750
                   18500
                   9375
                   91250
                    2500
                   887500
                      375
                   8871250
                        30
                   88712200
                          1
      1274551506) 8871219900 (  06=c
                  7647309036  (Remainder)
                  1223910864
```

Hence 6^{th} root of 17685 is 5.106

5 : 7^{TH} ROOT

Example 5.1 : Find 7^{th} root of 170859375

$a^7 = 1^7$; $1 = a$

$7a^6 = 7.1^6 = 7$

$21\,a^5.b^2 = 21.1^5.5^2 = 525$

$35a^4b^3 = 3^5.\,1^4\,.\,5^3 = 4375$

```
      1 7 0 8 5 9 3 7 5
      1                    1 = a
   7) 160 (5=b
       35
      1258
       525
      7335
      4375
```

	29609
$35a^3b^4 = 35 \, . \, 1^3 \, . \, 5^4 = 21875$	21875
	77343
$21\,a^2b^5 = 21 \, . \, 1^2 \, . \, 5^5 = 65625$	65625
	117187
$7ab^6 = 7. \, 1 \, . \, 5^6 = 109375$	109375
	78125
$b^7 = 5^7 = 78125$	78125
	×

Hence 7th root of 170859375 is 15.

n^{th} ROOT

Methods discussed above may be extended to obtaining n^{th} root (wherein n is any positive integer) of a number, say N. Subdivide the digits of N into blocks of n digits by putting a dash on the digit at the first place (i.e. at the unit place) and a dot on each of the next (n-1) digits. Continue this process. The last block, which is our Pada I, might not consist of n digits. Now subtract from the Pada I the n^{th} power of the greatest possible digit. Now form Pada II by associating the first digit of the last but one block with the remainder. Now divide Pada II by $na^{n-1} = {}^nC_1a^{n-1}$* . Assume that the quotient is b. form Pada III by associating the next digit of the last but one block with the remainder obtained on dividing by na^{n-1}. Now subtract from Pada III ${}^nC_2a^{n-1}b$. Continue this process of subtraction from the subsequent Padas up to ${}^nC_nb^n$. If the digits of N have not been exhausted, continue the whole process starting with $n(10a+b)^{n-1}$ instead of a . See Example 2.2.

Indeed the method is some what an inversion of the Binomial Theorem:

$$(a+b)^3 = a^3 + 3a^2b+3ab^2+b^3$$

$$(a+b)^4 = a^4+ 4a^3b+6a^2b^2+4ab^3+b^4$$

$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3 + 5ab^4+b^5$$

* ${}^nC_r$ stands for $n! / r! \, (n-r)!$

....

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC\, a^{n-2} b^2 + \ldots + {}^nC_n + b^n$$

In order to make the 'inversion process' more tangible, we obtain below 6[th] root of an algebraic expression. For the sake of simplicity we find out 6[th] root of ($a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6$). The following is the calculation chart:

$$a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6$$

$$= a$$

$$\begin{array}{lc}
 & a^6 \\
6a^5 &)\,6a^5b\,(\,b \\
 & \underline{6a^5b} \\
 & 15a^4b^2 \\
15a^4b^2 & \underline{15a^4b^2} \\
 & 20a^3b^3 \\
20a^3b^3 & \underline{20a^3b^3} \\
 & 15a^2b^4 \\
15a^2b^4 & \underline{15a^2b^4} \\
 & 6ab^5 \\
6ab^5 & \underline{6ab^5} \\
 & b^6 \\
b^6 & \underline{b^6} \\
 & \times
\end{array}$$

$$(a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6ab^5 + b^6)^{1/6} = a+b$$

Note that in this algebraic expression 6[th] root will be a+b and not 10a+b since a,b are not restricted to digits only.

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प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका J. Natur. Phys. Sci. Vol. 3 (89), 90-104

PRESERVATION OF GENERALIZED CONTINUITY

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ABSTRACT

The main purpose of this paper is to study the effects of uniform convergence and its various generalizations on functions which are nearly-continuous or quasi-continuous. It is known that uniform convergence preserves near continuity as well as quasi-continuity. In this paper a general result is proved to the effect that uniform convergence preserves several weaker forms of continuity including near and quasi continuity. Investigations are also made concerning Dini convergence, Leader convergence, Simple uniform convergence etc. in relation to functions which are nearly or quasi continuous. Necessary tools in the form of pretopology and preproximity are also developed.

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KEY WORDS AND PHRASES

Uniform convergence, Arzelà convergence, Dini convergence, Leader convergence, Quasi-uniform convergence, Simple uniform convergence, Simple Leader convergence, Semi-R_0, Semicompact, Pretopology, Preproximity, Near c** ontinuity, Quasi-continuity, Correct uniformity.

1. INTRODUCTION

Among the various generalizations of continuity, besides upper and lowr semicontinuity, two stand out viz.. *near continuity* (which Blumberg [3] called *densely approached*) and *quasi-continuity* (Kempisty [9]). Near continuity occurs naturally in problems involving open mapping and closed graph theorems of Functional Analysis [13]. On the other hand quasi-continuity is at the basis of results involving joint versus separate continuity. Hence it is no wonder that these two generalizations of continuity have been rediscovered several times under differenent names! In this paper we propose to study the effects of various convergences, which are generalizations of uniform convergence, on these two weak forms of continuity. For a study of continuous functions the reader is referred to (Di Concilio-Naimpally [4]).

Suppose (X, τ) and (Y, σ) are nonempty topological spaces. When we wish to discuss uniform convergence or generalisations we naturally assume that σ is Tychonoff and has a compatible (generalized) uniformity as well as a compatible EF proximity δ_2.

(1.1) *Definition* : A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is **quasi continuous** at $x_0 \in X$ iff for each $U \in \tau$, $V \in \sigma$, $x_0 \in U$, $f(x_0) \in V$ there is a nonempty set $W \in \tau$ such that $W \subset U$ and $f(W) \subset V$. Clearly f is quasi continuous iff for each $V \in \sigma$, $f^{-1}(V) \subset f^{-1}(V)^{0-}$. where 0 and - are interior and closure operators respectively (Kempisty [19]).

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(1.2) *Definition* : A function $f : (X, \tau) \to (Y, \sigma)$ is **nearly continuous** at $x_0 \in X$ iff for each $V \in \sigma$ containing $f(x_0)$, there is a nbhd. U of x_0 such that $U \subset f^{-1}(V)^-$. Obviously f is nearly continuous iff for each $V \in \sigma$, $f^{-1}(V)\, f^{-1}(V)^{-0}$.

Although there is a vast literature on quasi continuity and near continuity, it is not widely known that they together characterize continuity when (Y, σ) is regular.

(1.3) *Theorem.* (Garg-Naimpally [6]) If $f : (X, \tau) \to (Y, \sigma)$ is quasi continuous and nearly continuous then f is θ-continuous. Hence if (Y, σ) is regular, then f is continuous if and only if f is quasi-continuous and nearly continuous

Now we show how quasi continuous and nearly continuous functions can be considered as continuous when the topology τ on the domain X is replaced by larger families called *pretopologies*. From the definitions (1.1) and (1.2) it is clear that the following families are associated with quasi continuous and nearly continuous functions respectively :

(1.4) $$\tau_q = \{A \subset X : A \subset A^{0-}\}$$

(1.5) $$\tau_n = \{A \subset X : A \subset A^{-0}\}$$

It is easy to check that τ_q and τ_n both contain τ and are closed under *arbitrary unions* but not necessarily under *finite intersections*. This leads us to the following definition :

(1.6) *Definition* : A family of subsets ρ of X is a **pretopology** iff ρ contains ϕ, X and is closed under arbitrary unions. (X, ρ) is called a **pretopological space.**

(1.7) *Remarks*. It is now obvious that

(a) $f : (X, \tau) \to (Y, \sigma)$ is quasi continuous iff $f : (X, \tau_q) \to (Y, \sigma)$ is continuous.

(b) $f : (X, \tau) \to (Y, \sigma)$ is nearly continuous iff $f : (X, \tau_n) \to (Y, \sigma)$ is continuous.

We now show that uniform convergence preserves continuity even when the domain space is pretopological. This result includes, as special cases, the results of Bledsoe [2] concerning neighborly i.e. quasi-continuous functions and Husain [8] concerning almost i.e. nearly continuous functions.

(1.8) *Theorem.* (Garg-Naimpally) Suppose (X, ρ) is a pretopological space and (Y, ν) an AN uniform space. If a net of continuous functions $\{f_n : n \in D\}$ converges uniformly to a function f, then f in continuous.

Proof. (We may suppose that all entourages are symmetric in this and all subsequent proofs.) For $V \in \nu$ there is a $V_1 \in \nu$ such that $V_1^3 \subset V$. Since $f_n \to f$, there is an $m \in D$ such that for all $n > m$ and for each $x \in X$, $f_n(x) \in V_1[f(x)]$. Since each f_n is continuous at $x_0 \in X$, there is $U \in \tau$ such that $x_0 \in U$ and $f_n(U) \subset V_1[f_n(x_0)]$ where we may suppose that $n > m$ then $f(U) \subset V_1^3[f(x_0)] \subset V\{fx_0)]$. Hence f is continous at x_0.

(1.9) *Corollary.* Uniform convergence preserves quasi-continuity (Bledsoe [2]) and near continuity (Husain [8]).

2. PRETOPOLOGY AND PREPROXIMITY.

If (X, ρ) is a pretopological space, then a set C is *closed* iff $X - C \in \rho$. We now define a *preclosure operator* $\text{pcl} : P(X) \to P(X)$:

(2.1) *Definition*: $\text{pcl}(A) = \cap \{C : A \subset C, X - C \in \rho\}$.

The following result is obvious :

(2.2) *Theorem* : The preclousre opertaor pcl on a pretopological space (X, ρ) satisfies:

(a) $\text{pcl}(\phi) = \phi$,

(b) $A \subset \text{pcl}(A)$ for each $A \subset X$,

(c) $\text{pcl}(\text{pcl}(A)) = \text{pcl}(A)$,

(d) $A \subset B$ implies $\text{pcl}(A) \subset \text{pcl}(B)$.

(2.3) *Remarks*. Of course, pcl does not satisfy the union axiom but it satisfies the weaker axiom (d) which is equivalent to (d') $\text{pcl}(A) \cup \text{pcl}(B) \subset \text{pcl}(A \cup B)$.

The next result is easy to prove.

(2.4) *Lemma*. Let (X, ρ) be a pretopological space. Then, for each $A \subset X$, $\text{pcl}(A) = \{x \in X : \text{for each } U \in \rho,\ x \in U \Rightarrow U \cap A \neq \phi\}$

Furthermore, every preclosure oeprator pcl satisfying 2.2 (a) - (d), induces a pretopology ρ' where

$\rho' = \{U : \text{pcl}(X - U) = X - U\}$.

Notation. $\underline{A} = \text{pcl}(A)$.

We now define a preproximity.

(2.5) *Definition* : A binary relation δ on $P(X)$ is called a **basic preproximity** iff it satisfies.

(a) $A \,\delta\, B$ implies $B \,\delta\, A$,

(b) $A \,\delta\, B$ implies $A \neq \phi$, $B \neq \phi$,

(c) $A \cap B \neq \phi$ implies $A \,\delta\, B$,

(d) $A \,\delta\, B$ and $B \subset C$ implies $A \,\delta\, C$

A basic preproximity δ is called an **S-preproximity** if it satisfies :

(S) $a \,\delta\, B$ and $b \,\delta\, C$ for each $b \in B$, then $a \,\delta\, C$.

A basic preproximity δ is a **LO-Preproximity** if it satisfies:

(LO) $A \delta B$ and $b \delta C$ for each $b \in B$ implies $A \delta C$.

A basic preproximity is an **R-preproximity** if it satisfies :

(R) $a \not\delta B$ implies there is an $E \subset X$ such that $a \not\delta E$ and $(X - E) \not\delta B$.

(2.6) *Theorem* : The basic preproximity δ on X induces an operator δ on P(X) to P(X):

(2.7) $$A^{\delta} = \{ x \in X : x \delta A \}$$

The above defines a preclosure operator if an only if δ is an S-preproximity. The pretopology induced by a preproximity δ is denoted by $\rho(\delta)$. We say that δ and $\rho(\delta)$ are **compatible.**

(2.8) *Definition* : A pretopological space (X, ρ) is **(symmetric) R_0** iff $x \in \underline{y}$ implies $y \in \underline{x}$.

The following results are proven in a similar manner to the corresponding results on proximities and topological spaces [11].

(2.9) *Theorem*. Every separated LO preproximity space (X, δ) induces a T_1 pretopological space $(X, \rho(\delta))$. Conversely, every T_1 pretopological space (X, ρ) has the finest compatible LO- preproximity δ_0 as well as the coarsest compatible LO- preproximity δ_{CL} where

(2.10) $$A \delta_0 B \text{ iff } \underline{A} \cap \underline{B} \neq \phi ,$$

(2.11) $$A \delta_{CL} B \text{ iff } A \delta_0 B \text{ or both } \underline{A}, \underline{B} \text{ are infinite.}$$

(2.12) *Theorem*. If δ is a compatible LO-preproximity on a pretopological space (X, ρ), then

$$A \delta B \text{ iff } \underline{A} \delta \underline{B}$$

(2.13) *Definition.* A pretopological space (X, ρ) is **regular** iff for each x not in a preclosed set C, there are $U, V \in \rho$, $x \in U$, $C \subset V$ and $U \cap V = \phi$.

(2.14) *Theorem.* A pretopological space (X, ρ) is regular if and only if it has a compatible LO-R preproximity. The coarsest compatible R-S-preproximity δ_{CS} is given by

(2.15) $A \, \delta_{CS} \, B$ iff $(A \cap \underline{B}) \cup (\underline{A} \cap B) \neq \phi$. or A, B are both infinite.

The finest compatible S preproximity δ_S is given by

(2.16) $\quad A \, \delta_S \, B$ iff $(\underline{A} \cap B) \cup (A \cap \underline{B}) \neq \phi$.

3. **QUASI-CONTINUITY.** In this section we pursue the limits of nets of quasi-continuous functions under various convergences. If (X, τ) is a topological space, then members of τ_q (1.4) are called **semi-open** and the resulting preclosure operator is called scl (**semi-closure operator**). In this section $\underline{A}$ denotes scl (A) and (X, τ) is called **semi - R_0** iff (X, τ_q) is R_0 (2.8). Such a space has compatible LO-preproximities defined by (2.10), (2.11) and if (X, τ_q) is regular then it has compatible S-preproximities defined by (2.15), (2.16).

In this section (X, τ) is a semi-R_0 topological space and δ_1 is any preproximity compatible with (X, τ_q). (Y, σ) dentoes an R_0 topological space with a compatible proximity δ_2. $(f_n : n \in D)$ denotes a net of functions on X to Y.

(3.1) *Definition.* A function $f : (X, \delta_1) \to (Y, \delta_2)$ is **p-quasi-continuous** iff whenever $A \, \delta_1 \, B$ in X, $f(A) \, \delta_2 \, f(B)$ in Y.

(3.2) *Remarks.* It is easy to see that a p-quasi-continous function is quasi-continuous. The following example shows that the converse is not true. Suppose $X = Y = [-1, 1]$ with the usual topology τ. Let δ_2 be the

usual metric proximity on Y and δ_1 the coarsest preproximity δ_{CL} defined by (2.11).

$$f(x) = \begin{cases} 0 & at\ x = 0 \\ \sin 1/x & at\ x \neq 0 \end{cases}$$

Then f is quasi-continuous but not p-quasi-continuous. If $A = \{(n\pi)^{-1} : n \in N\}$, $B = \{[(2n+1)\pi/2]^{-1} : n \in N\}$, then A δ_1 B but f(A) δ_2 f(B).

For definitions of various convergences such as R.C., L.C., D.C., S.U.C. Q.U.C., S.L.C. and A.C. please see [4]. Here we merely give examples and state results which are not in [4].

(3.3) *Theorem* : If $f_n \xrightarrow{R.C.} f$ and each f_n is (p-) quasi-continuous, then f is (p-) quasi-continuous.

(3.4) *Example*. Let X = Y = R with the usual topology. For each $x \in X$ and $n \in N$ we set:

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$
$$f_n(x) = f(x) + n.$$

Then f_n and f are quasi-continuous functions, $f_n \xrightarrow{R.C.} f$ but $f_n \xrightarrow{P.C.} f$. Since P.C. does not necessarily preserve quasi-continuity (Levine [10]), it follows that P.C. and R.C. are independent.

(3.5) *Corollary*, X.C. preserves (p-) quasi-continuity.

(3.6) *Theorem* : If δ_2 is EF and $f_n \xrightarrow{L.C.} f$ then $f_n \xrightarrow{X.C.} f$; hence LC. preserves (p-) quasi-continuity.

(3.7) *Example*. Suppose X = Y = R with the usual topology. For each $x \in X$ and $n \in N$ we set :

$$f(x) = \begin{cases} x^2 - 1 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

$$f_n(x) = \left(f(x)+\frac{1}{n}\right)^2$$

Clearly $f_n \xrightarrow{P.C.} f$ but $f_n \not\xrightarrow{L.C.} f$. for $A = N-\{1\}$, $B = \{(n+1/n)^2 : n \in N]$, $f(A) \not\delta_2 B$ but $f_n(A) \cap B \neq \phi$ for each $n \in N$ and so $f_n(A) \delta_2 B$. If δ_2 is the coarsest compatible S-proximity on R, then $f_n \xrightarrow{L.C.} f$. and $f_n \xrightarrow{X.C.} f$. Furthermore, if δ_2 is the usual EF proximity, then $f_n \xrightarrow{X.C.} f$.

(3.8) *Example* : Suppose X= Y= [-1, 1] with the usual topoloyg δ_1 the finest compatible preproximity (2.10) on (X, τ_q) and δ_2 the unique compatible EF proximity on Y. For each $x \in X$, $n \in N$ we define :

$$f(x) \equiv 0$$

$$f_n(x) = \begin{cases} 0 & x = 0 \\ \dfrac{n \sin(1/x)}{1+n^2 \sin^2(1/x)} & \text{otherwise} \end{cases}$$

Here $f_n \xrightarrow{P.C.} f$. and $f_n \xrightarrow{R.C.} f$ and so $f_n \xrightarrow{X.C.} f$. But there exists a sequence $(x_n) \in X$ such that $\sin(1/x_n) = {}^1/_n$. Set $A=\{x_n\}$, $B = \{1/2\}$; then $f(A) \not\delta_2 B$ but $f_n(A) \delta_2 B$ for each $n \in N$. Thus $f_n \not\xrightarrow{L.C.} f$. .

(3.9) *Example* : The example (2.9) (c) of Di Concilio-Naimpally [4] shows that the sequence

$$f_n(x) = \begin{cases} x + 2/n & x \leq 0 \\ x + 1/n & x > 0 \end{cases}$$

is a squence of quasi-continuous functions which L.C. converges to the continuous function f(x) = x in the usual proximity. But if δ_r is the R-proximity defined by

(3.10) $\qquad A \delta_r B$ iff $(A \cap B^-) \cup (A^- \cap B) \neq \phi$.

then $f_n \not\xrightarrow{X.C.} f$. and $f_n \not\xrightarrow{L.C.} f$.

(3.11) *Theorem* : If (X, τ) is semi-compact (i. e) every semi-open cover of X admits a finite subcover [14] and $f_n \xrightarrow{S.\,L.\,C.} f$ which is quasi-continuous, then $f_n \xrightarrow{Q.\,U.\,C.} f$.

(3.12) *Theorem* : Suppose X is semi-compact and δ_2 is EF. If $f_n \xrightarrow{P.\,C.} f$ and each f_n and f are quasi-continuous, then $f_n \xrightarrow{A.\,C.} f$.

Proof. Suppose f is quasi-continuous, $p \in X$, $B \subset X$, and $f(p) \not\delta_2 f(B)^-$. Since f is quasi-continuous, $f(\underline{B}) \subset f(B)^-$ and since X is semi-compact we may suppose that B is semi-compact. Since δ_2 is EF, there is an open set $V \subset Y$ such the $f(B) \subset V$ and $f(p) \not\delta_2 V$. Suppose $m \in D$. Since $f_n \xrightarrow{P.\,C.} f$., there exists an $m' > m$ such that $f_n(p) \not\delta_2 V$ for each $n > m'$. Also we have $B \subset f_{n_i}^{-1}(V)$ for each $n > m'$. Since B is semi-compact and each $f_n^{-1}(V)$ is semi-open, $B \subset \bigcup_{i \in I_m} f_{n_i}^{-1}(V)$. Consequently,

$$B = \bigcup_{i \in I_m} B_i \text{ where } B_i = B \cap f_{n_i}^{-1}(V).$$

Hence $f_{n_i}(p) \not\delta_2 f_{n_i}(B_i)$ i.e. $f_n \xrightarrow{A.\,C.} f$.

(3.13) *Theorem* : If X is semi-compact and extremally disconnected, δ_2 if EF, $f_n \xrightarrow{A.\,C.} f$, each f_n is quasi-continuous implies f is quasi-continuous.

Proof. In this case the semi-closure operator (Njastad [12]) is a Kuratowski closure operator and we follow the proof of Di Concilio-Namipally [4].

4. NEAR CONTINUITY : In this section we study the convergence of nearly continuous functions. Since several of the results of the last section go through easily for nearly continuous functions as well, we mostly present counter examples in this section.

If (X, τ) is a topological space, then τ_n as defined by (1.5) consists of nearly open sets [13]. Complements of the members of τ_n are called **nearly closed** and they naturally induce a preclosure operator ncl. We write $\underline{A}$ = ncl (A). Suppose (X, τ) is such that ncl $\{x\} = \{x\}$ for each $x \in X$. Then (X, τ_n) has a compatible LO preproximity δ_0 defined by

$$A \, \delta_0 \, B \text{ iff } \operatorname{ncl}(A) \cap \operatorname{ncl}(B) \neq \phi$$

If (Y, σ), is a topological space, then $f : (X, \tau) \rightarrow (Y, \sigma)$ is nearly continuous iff

$$f(\underline{A}) \subset f(A)^- \text{ for each } A \subset X.$$

Definition of a p-nearly continuous function is similar to that of a p-quasi continuous function.

4.1*Example*. We now present an example of a nearly continuous function which is not p-nearly continuous. $X = Y = R$ with the usual topology.

$$f(x) = \begin{cases} 0 & x \in Q \\ 1 & \text{otherwise} \end{cases}$$

If δ_{CL} denotes the coarsest compatible LO preproximity on (X, τ_n), then

$A \, \delta_{CL} \, B$ but $f(A) \not\delta_2 f(B)$

Where $A = \{n : n \in N\}$, $B = \{n + \sqrt{2} : n \in N\}$ and δ_2 is the usual metric proximity on Y.

(4.2) *Definition*. A topological space (X, τ) is strongly compact iff every nearly open cover of X has a finite subcover [1].

Results analogous to (3.3) and (3.6) are ture for nearly continuous functions. Analogue of Theorem (3.11) also holds if we replace semi-

compact by strongly compact. Similarly semi-compact is to be replaced by strongly compact in (3.12).

(4.3) *Lemma*. (Ganster [5]) Let $X = F \cup G$ be the Hewitt representation of X, where F is closed and resolvable, G is open and hereditarily irresolvable with $F \cap G = \phi$. The following are equivalent,

(a) (X, τ_n) is a topology

(b) G is open and each point of F^o is nearly open.

Analogue of Theorem (3.13) is true if we replace semi-compact by strongly compact and extremally disconnected by (a) or (b) of Lemma (4.3).

We now present counterexamples :

(4.4) *Example*. X = Y = R with the usual topology.

$$f(x) = \begin{cases} 0 & x \in Q \\ 1 & \text{otherwise} \end{cases}$$

$f_n(x) = f(x) + n.$

Then $f_n \xrightarrow{R.C.} f$ but $f_n \not\xrightarrow{P.C.} f$. Since Husain [8] has shown that P.C. does not preserve near continuity, we have P.C. and R.C. independent.

(4.5) *Example* . X = Y= [-1, 1] with the usual topology.

$f(x) = 0$

$$f_n(x) = \begin{cases} 1/n & x \in X - Q \\ nx/(1 + n^2x^2) & \text{otherwise.} \end{cases}$$

Then $f_n \xrightarrow{P.C.} f$., $f_n \xrightarrow{R.C.} f$ but $f_n \not\xrightarrow{L.C.} f$, since if $A = \{1/n : n \in N\}$; $B = \{1/2\}$, $f(A) \bar{\delta}_2 B$ but $f_n(A)\ \delta_2 B$.

(4.6) *Example*. X =Y= R with the usual topology.

$$f(x) = \begin{cases} x^3 & x \in Q \\ -x^3 & \text{otherwise} \end{cases}$$

$$f_n(x) = \begin{cases} \left(x + \frac{1}{n}\right)^3 & x \in Q \\ -\left(x + \frac{1}{n}\right)^3 & \text{otherwise.} \end{cases}$$

Set $A = \{n \in N - \{1\}\}$, $B = \{n + 1/n)^3 : n \in N - \{1\}\}$. Then $f(A) \not\delta_2$ B but $f_n(A) \cap B \neq \phi$ for each $n \in N$. Consequently, $f_n \not\xrightarrow{L.C.} f$. But if $\not\delta_2$ is the R-proximity defined by (3.10), then $f_n \xrightarrow{L.C.} f$. However, $f_n \xrightarrow{X.C.} f$ if δ_2 is EF or the above.

(4.7) *Example* . X = Y = R with the usual topology.

$$f(x) = \begin{cases} x & x \in Q \\ 0 & \text{otherwise} \end{cases}$$

$$f_n(x) = \begin{cases} x + 1/n & x \in Q - \cup \{O\} \\ x + 2/n & x \in Q + \\ 0 & \text{otherwise.} \end{cases}$$

If δ_2 is the usual metric proximity, then $f_n \xrightarrow{L.C.} f$. But if $\not\delta_2$ is (3.10), then $f_n \not\to f$ in X.C. or L.C.

CONCLUSION : We have already noted that the family of nearly open sets or the family of semi-open sets does not form a topology for R. We recall the R with the usual topology is anti-semicompact i.e. every semicompact subset is finite [14]. Furthermore, no closed interval of R can be strongly compact. Hence we cannot have Arzela-type convergence for quasi-continuous or nearly continuous functions on a closed bounded interval of R.

The techniques developed in this paper can also be used to study the convergence of other generalized forms of continuity viz. *weak continuity*, *complete continuity*, *α-continuity*, *supercontinuity* (for definitions see [7]). In fact it is not difficult to set up, for each of these forms of generalized continuity, an appropriate pretopology and an associated LO-preproximity on the domain. Then we can show that these generalized forms of continuity are preserved under U.C., D.C., L.C. and X. C. It is also not difficult to see that U.C. preserves feeble continuity [7].

However, to attack the problem of preservation of *strong continuity*, *irresoluteness and* *δ-Continuity* [7], we are forced to set a pre-topology and a pre-proximity on both the domain and the range. In this manner, we can show that irresoluteness, strong continuity and δ-continuity are preserved only under X.C. and D.C.

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A NOTE ON THE STRESS DISTRIBUTION IN A ROTATING CIRCULAR DISK OF NONHOMOGENEOUS MATERIAL UNDER TRANSIENT SHEARING FORCE

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ABSTRACT

In this note, distribution of stress in a nonhomogeneous annular circular plate rotating with time dependent angular velocity has been found. Numerical results have been presented to study the effect of nonhomogeneity on stress components

INTRODUCTION

The problem of stress distribution in a steadily rotating circular disk has been gaining considerable interest among the research workers for its applications in engineering fields. In literature we find many investigators concentrating their attention on the problem under different conditions. The works of Singh and Puri [1], Samanta [2], Chakravorty and Chaudhuri [3] may be referred in this connection. In the present note my aim is to study the problem assuming the material of the disk as nonhomogeneous in the sense that both the Young's modulus E and the density ρ of the material are functions of position. It is also assumed that the disk is rotating with exponentially decaying time-dependent angular velocity. The problem has been completely solved and the expressions for the stresses have been given in terms of infinite series. Finally, to study the effect of

nonhomogeneity on the stresses numerical computations have been done and the results have been graphically exhibitted.

FORMULATION OF THE PROBLEM

We assume that the stresses and displacements do not vary across the thickness of the disk and the lateral surface is free from any stress. Taking the centre of the circular plate as pole and using polar coordinates (r, θ) we find that for reasons of symmetry, the radial displacement U_r and the tangential displacement U_θ are independent of θ.

The equations of motion are

$$\frac{\partial \sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}+\rho\,\Omega^2 r=\rho\,\frac{\partial^2 U_r}{\partial t^2} \tag{1}$$

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}-\rho\,\frac{d\Omega}{dt}r=\rho\,\frac{\partial^2 U_r}{\partial t^2}$$

where σ_r, σ_θ and $\tau_{r\theta}$ are the radial, tangential and the shearing stress components respectively given by

$$\sigma_r=\frac{E}{1-\upsilon^2}\left[\frac{\partial U_r}{\partial r}+\nu\,\frac{U_r}{r}\right]$$

$$\sigma_\theta=\frac{E}{1-\upsilon^2}\left[\upsilon\,\frac{\partial U_r}{\partial r}+\frac{U_r}{r}\right] \tag{2}$$

$$\tau_{r\theta}=\frac{E}{2(1+\upsilon)}\left[\frac{\partial U_\theta}{\partial r}-\frac{U_\theta}{r}\right]$$

E = E (r) is the Young's modulus, ρ = ρ (r) is the variable density, ν is the Poisson's ratio of the material and Ω = Ω (t) is the angular velocity of the disk. We assume that Ω varies exponentially with time such that

$$\Omega=\Omega_0 e^{-p\tau}$$

where Ω_0 and $p\ (>O)$ are constants and $\tau\ \ t/t_0$

The boundary conditions are

$$\tau_{r\theta} = -M e^{-pt} \qquad on\ r = a\ (M\ is\ a\ constant)$$

$$= 0 \qquad on\ r = b$$

$$\sigma_r = 0 \qquad on\ r = a\ and\ r = b$$

where a and b are the outer and inner radii of the annular disk. The non-homogeneity of the material of the disk is assumed to obey the laws

$$E = E_0 e^{\beta x}$$

$$\rho = \rho_0 e^{\beta x}$$

where $x = r/b$ and E_0 and ρ_0 are constants and β is a real number.

Substitutions from (2) and (5) into (1), leads to two coupled differential equations in U_r and U_θ. The nature of angular velocity Ω in (3) suggests U_r and U_θ in the following forms

$$U_r = F_1(x)\, e^{-2p\tau/b} \tag{6}$$

$$U_\theta = F_2(x)\, e^{-p\tau/b}$$

With such a choice, the field equations in non-dimensional variable x may be written as

$$\frac{d^2F_1}{dx^2} + \left(\beta + \frac{1}{x}\right)\frac{dF_1}{dx} + \left(\frac{\upsilon\beta}{x} - \frac{1}{x^2} - K_2\right)F_1 = -K_1 x \tag{7}$$

and

$$\frac{d^2F_2}{dx^2} + \left(\beta + \frac{1}{x}\right)\frac{dF_2}{dx} - \left(\frac{\beta}{x} + \frac{1}{x^2} + K_4\right)F = -K_3 x \tag{8}$$

where

$$K_1 = \rho_o \ \Omega_o^2 b^2 (1-\upsilon^2)/E_o$$

$$K_2 = 4 \ \rho_o \ p^2 \ b^2 (1-\upsilon^2)/E_o \ t_o^2$$

$$K_3 = 2 \ \rho_o \ \Omega_? \ b^2 p \ (1+\upsilon)/E_o \ t_o$$

$$K_4 = 2 \ \rho_o \ b^2 p^2 \ (1+\upsilon)/E_o \ t_o^2$$

Thus our problem is to solve (7) and (8) under boundary conditions (4).

Solution of the problem

Setting $F_1 = x. V_1$

and $F_2 = x. V_2$, equations (7) and (8) become

$$x \frac{d^2 V_1}{dx^2} + (3+\beta x)\frac{dV_1}{dx} + \left[\beta (1=\upsilon) - K_2 x\right] V_1 = -K_1 x \qquad (9)$$

$$x \frac{d^2 V_2}{dx^2} + (3+\beta x)\frac{dV_2}{dx} - (K_4 x) V_2 = -K_3 x \qquad (10)$$

The complementary equations of (9) and (10) may be easily transformed into confluent hypergeometric differential equations by suitable substitutions. Since the solution of such equation is known the general solution of (9) (10) may be given by

$$F_1 = x.\, e^{s_1 x}\left[A_1 \phi_1 (C_1 x) + B_1 \psi_1 (C_1 x)\right] + x\eta_1 (x) \qquad (11)$$

and

$$F_2 = x.\, e^{s_2 x}\left[A_2 \phi_2 (C_2 x) + B_2 \psi_2 (C_2 x)\right] + x\eta_2 (x) \qquad (12)$$

Where ϕ_1, ψ_1 and ϕ_2, ψ_2 are respectively known solutions of the confluent hypergeometric equations, [4]

$$\xi \frac{d^2 y}{d\xi^2} + (3-\xi)\frac{dy}{d\xi} - \alpha_1 y = 0$$

and

$$\xi \frac{d^2y}{d\xi^2} + (3-\xi)\frac{dy}{d\xi} - \alpha_2 y = 0$$

Where

$\alpha_1 = (3s_1 + \beta(1+\upsilon))/[2s_1+\beta], C_1 = -(2s_1+\beta)$

$\alpha_2 = (3s_2/[2s_2+\beta], C_2 = -(2s_2+\beta)$

s_1 *is a root of* $\theta^2 + \beta\theta - K_2 = 0$, *while* s_2 *is that of* $\theta^2 + \beta\theta - K_4 = 0$

The functions η_1 (x) and η_2 (x) appearing in (11) and (12) are given by

$$\eta_1(x) = \sum_{n=2}^{\infty} a_n x^n$$

$$\eta_2(x) = +K_3/K_4$$

where

$a_2 = -K_1/8$

$a_3 = \beta(3+\upsilon).K_1/120$

and other a_i's can be calculated from the relation

$(n+1)(n+3)a_{n+1} + \beta(1+\upsilon+n)a_n - K_2 a_{n-1} = 0$ *for* $n \geq 3$.

Knowing F_1 and F_2, we can find U_r and U_θ and U_θ from (6). Using these U_r and U_θ, we get from (2)

$$\sigma_r = \frac{E_0 e^{-2p\tau}}{b(1-\upsilon^2)}\left[A_1R_1(x) + B_1R_2(x) + R_3(x)\right]$$

$$\sigma_r = \frac{E_0 e^{-2p r}}{b(1-\upsilon^2)}\left[A_1R_4(x) + B_1B_5(x) + R_6(x)\right]$$

and

$$\tau_{r\theta} = M \,.\, e^{-p\tau}\left[A_2 R_7\langle x\rangle + B_2 R_8(x)\right]$$

where

$$R_1(x) = [\,(1+\upsilon+s_1 x)\,\phi_1(C_1 x) + C_1 x\,\phi'_1(C_1 x)\,]\,e^{(\beta+s_1)x}$$

$$R_2(x) = [\,(1+\upsilon+s_1 x)\,\psi_1(C_1 x) + C_1 x\,\psi'_1(C_1 x)\,]\,e^{(\beta+s_1)x}$$

$$R_3(x) = [\,(1+\upsilon) + \eta_1(x) + x.\,\eta'_1(x)\,]\;e^{\beta x}$$

$$R_4(x) = [\,(1+\upsilon+\upsilon x s_1)\,\phi_1(C_1 x) + \upsilon x C_1 .\phi'_1(C_1 x)\,]\,e^{(\beta+s_1)x}$$

$$R_5(x) = [\,(1+\upsilon+\upsilon x s_1)\,\psi_1(C_1 x) + \upsilon x C_1 .\;\psi'_1(C_1 x)\,]\,e^{(\beta+s_1)}$$

x

$$R_6(x) = [\,(1+\upsilon)\,\eta_1(x) + x\upsilon\;\eta'_1(x)\,]\,e^{(\beta x)}$$

$$R_7(x) = [\,(x\;s_2)\,\phi_2(C_2 x) + .\,C_2\;\phi'_1(C_2 x)\,]\,e^{(\beta+s_2)x}$$

$$R_8(x) = [\,(x\;s_2)\,\psi_2(C_2 x) + .\,C_2\;\psi'_1(C_2 x)\,]\,e^{(\beta+s_2)x}$$

The constants A_1, B_1, A_2, B_2 *in (13), (14) and (15) may be* determined from the boundary conditions (4), as

$$A_1 = [R_2(1)\,R_3(x_o) - R_2(x_o)\,R_3(1)\,]\,/\,\Delta_1$$

$$B_1 = [R_1(x_o)\,R_3(1) - R_1(1)\,R_3(x_o)\,]\,/\,\Delta_1$$

$$A_2 = [+R_8(1)\,]\,/\,\Delta_2$$

$$B_2 = -R_7(1)\,]\,/\,\Delta_2$$

where $x_o = a/b$

$$\Delta_1 = R_1(1)\,R_2(x_o) - R_1(x_o)\,R_2(1)$$

$$\Delta_2 = R_7(1)\,R_8(x_o) - R_7(x_o)\,R_8(1)$$

Thus knowing A_1, B_1, A_2, B_2, we get the complete expressions for the components of stresses from (13),(14) and (15).

Numerical results :

To study the nature of stress distribution and also the effect of nonhomogeneity on the stresses we compute radial and tangential stress components along with shearing stress component at different positions in the disk. In our computation we have assumed $p = 1/2$, $\Omega_0 t_0 = 1$, $\upsilon = 1/3$ and $\rho_0 \Omega^2{}_0 b^2 = E_0$

We plot variations of

$$P = \left[b\left(1-\upsilon^2\right) e^{2p\tau} / E_0 \right] \sigma_r$$

$$Q = -\left[b\left(1-\upsilon^2\right) e^{2p\tau} / E_0 \right] \sigma_\theta$$

$$R = \left[e^{p\tau} / M \right] \tau_{r\theta}$$

with position in Figure 1, 2 and 3.

The results in the associated homogeneous case ($\beta = 0$) have been shown in broken lines. The effect of non-homogeneity is clear from the graph.

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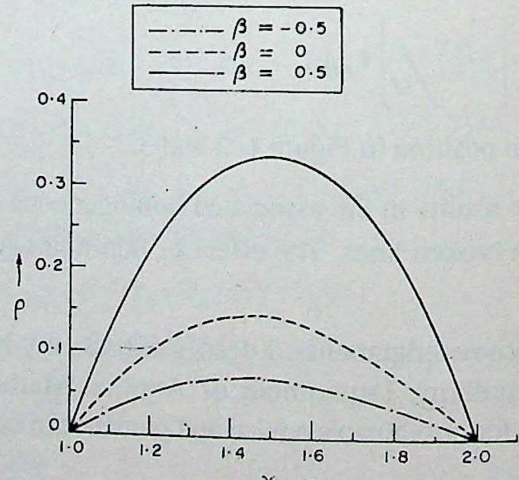

Fig. 1 Variation of radial stress with position (for fixed τ).

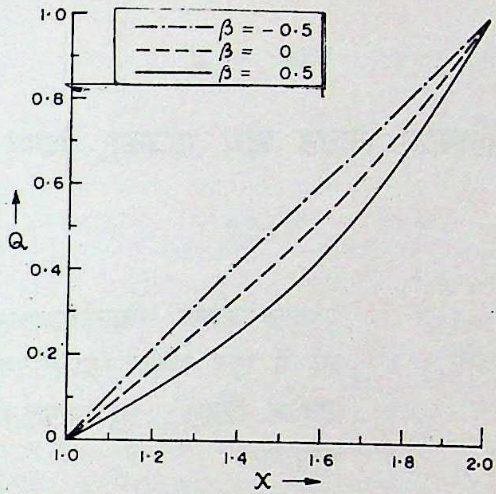

Fig. 2. Variation of shearing stress with position (for fixed τ).

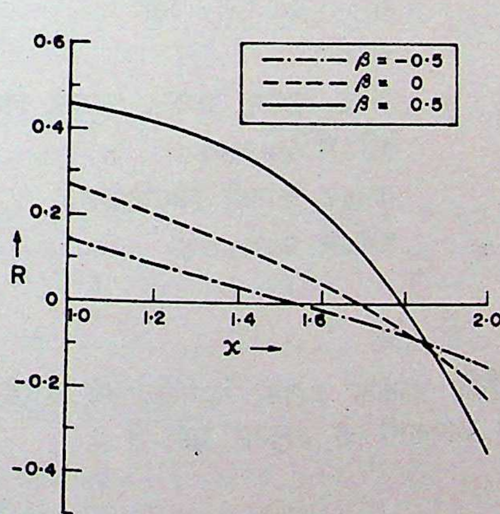

Fig. 3. Variation of tangential stress with position (for fixed τ).

फार्म – ४

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EXTENSIONS OF THE CARÍSTI-KIRD FIXED POINT THEOREM

Shyam L. Singh*

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ABSTRACT

Two coincidence theorems using an elementary constructive proof are proved in this paper which include *inter alia* a recent substantial generalization of the Caristi-Kirk fixed point theorem due to Bollenbacher and Hicks [1]. Serveral examples provide an insight into the results and raise new questions.

AMS (MOS) subject classifications (1880): 47H10, 54H25.

Keywords and phrases: Coincidence/fixed point theorem, Caristi-Kirk fixed point theorem, weakly preorbitally commuting maps, metric space.

1. INTRODUCTION

The Caristi-Kirk fixed point theorem [2] generalizes a number of fixed point theorems and has various proofs and several applications (see, for instance, [1]-[3], [5], [6]-[8], [10] and the referencesof [8]). However, Caristi's proof, as well as several others, lack iterative construction of the fixed point. A recent result of Bollenbacher and Hicks [1] (see also Hicks [5]) has an elementary constructive proof and improves the Caristi-Kirk theorem substantially.

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In this paper two coincidence theorems (Theorems 1-2) for a pair of maps on an arbitrary set with values in a metric space are given and the result of [1], as well as of others, may be derived as corellaries. Indeed, apart from obtaining the fixed point theorem of Bollenbacher and Hicks [1] exactly as a special case of Theorem 1 (Cf. Remark 1), a variant of their theorem is also obtained as a corellary (Corollary 3) of Theorem 2.

Throughout this paper, let A be an arbitrary set, (X, d) a metric space and f, T maps on A with values in X. A point z in A is said to be a coincidence point of f and T if fz = Tz. If for a point $x_o \in A$ there exists a sequence $\{x_n\}$ of points of A such that $fx_{n+1} = Tx_n$ $n=0,1,2....$, then

$$O(T,f;x_0) = \{fx_n : n = 1,2,.....\}$$

is said to be the orbit of (T,f) at x_0. We shall use $O(T,f;x_0)$ as a set and as a sequence as the situation demands. A subspace of X is said to be $(T,f;x_0)$-orbitally complete if the colsure of $O(T,f;x_0)$ is complete. Evidently, the completeness of X implies the orbital completeness, and the space may be $(T,f;x_0)$-orbitally complete without being complete (see, for instance, [11, Ex. 2.4]). A subspace of X will be called (T,f)-orbitally complete f it is $(T,f;x_0)$-orbitally complete for every $x_0 \in A$. $G:X \to (0,\infty)$ is $(T,f;x_0)$-orbitally lower semi-continuous (l.s.c.) if $\{z_n\}$ is a sequence in $O(T,f;x_0)$ and $\lim z_n=p$ implies $G(p) \leq \liminf G(z_n)$. If A =X and f=I (identity map) then G is said to be T-orbitally l.s.c. at x_0; [1], [8,p.62]. If A = X then: (i) T is said to be $(T,f;x_0)$-orbitally continuous [12] if the restriction of T on the closure of $O(T,f;x_0)$ is continuous; (ii) T and f

are said to be weakly commuting [9] iff $d(Tfx, fTx) \leq d(fx, Tx)$ for every x in $\mathbf{X}$; and (iii) T and f will be called weakly x_0-preorbitally commuting (w.x_0-p.c.) iff there exists a positive integer N such that $d(Tfx_n, fTx_n) \leq d(fx_n, Tx_n)$ for x_n (in $\geq N$) occurring in $O(T,f;x_0)$. Moreover, T and f will be called (T,f)-orbitally continuous (respectively weakly preorbitally commuting) if they are $(T,f;x_0)$-orbitally continuous (resp. w.x_0-p.c.) for ever $x_0 \in X$. We remark that commutativeity $\Rightarrow$ weak commutativety $\Rightarrow$ weak preorbital commutativity but not its vice-versa, (see Example 2, 5 below).

2. RESULTS

Consider the following condition for $\phi: \mathbf{X} \to [0, \infty)$ and every $y \subset A$:

(CP) $\qquad d(fy, Ty) \leq \phi(fy) - \phi(Ty).$

Park [6, Prop. 7] using a non-constructive proof shows the existence of a coincidence point of T and f under this condition when $A = X$ is complete, ϕ l.s.c. and f surjective. The Caristi-Kirk fixed point theorem [2] states that T satisfying (CP) with $A = \mathbf{X}$ complete, $f = I$ and ϕ l.s.c. has a fixed point. In the following Coincidence Theorems 1-2, assume that there exists an orbit for (T,f).

THEOREM 1. Let T, f be maps on A with values in X and $\phi : X \to (0,\infty)$. Suppose there exists an $x_0 \in A$ such that (CP) for every $fy \in O(T,f;x_0)$, and $f(A)$ is $(T,f;x_0)$-orbitally complete. Then:

(1.1) $\lim Tx_n = \lim fx_n = fz$ exists for some $z \in A$;

(1.2) $d(Tx_n, fx) \leq \phi(Tx_n)$;

(1.3) $Tz = fz$ iff $G(fz) = d(fz, Tz)$ is $(T, f; x_0)$-orbitally i.s.c.;

(1.4) $d(Tx_n, Tx_0) \leq \emptyset(Tx_0)$ and $d(fz, Tx_0) \leq \emptyset(Tx_0)$.

PROOF. Since
$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(fx_{n+1}, Tx_{n+1}) \\ &\leq \emptyset(fx_{n+1}) - \emptyset(Tx_{n+1}) \\ &= \emptyset(Tx_n) - \emptyset(Tx_{n+1}), \end{aligned}$$

that is

$$\sum_{k=0}^{n} d(Tx_k, Tx_{k+1}) \leq \emptyset(Tx_0) - \emptyset(Tx_{n+1}) \leq \emptyset(Tx_0)$$

$\{Tx_n\}$ and $\{fx_n\}$ are Cauchy sequences and both have the same limit. Call it p. Since $p \in f(A)$, there exists a z in A such that $fz = p$. This proves (1.1).

(*) $$0 \leq d(Tx_n, Tx_m) \leq \sum_{k=n}^{m-1} d(Tx_k, Tx_{k+1}) \leq \emptyset(Tx_n) - \emptyset(Tx_m) \leq \emptyset(Tx_n).$$

Making $m \to$ gives (1.2).

The $(T, f; x_0)$-orbital l.s.-continuity of G implies

$$\begin{aligned} 0 \leq d(fz, Tz) = G(fz) &\leq \lim\inf G(fx_n) \\ &= \lim\inf d(fx_n, Tx_n) \\ &= \lim\inf d(fx_n, fx_{n+1}) = 0. \end{aligned}$$

So $fz = Tz$.

If $Tz = fz$ and $\{fx_n\} \subset O(T, f; x_0)$ with $fx_n \to fz$ then $G(fz) = d(fz, Tz) = 0 \leq \lim\inf d(fx_n, fx_{n+1}) = \lim\inf d(fx_n, Tx_n) = \lim\inf G(fx_n)$. This completes the proof of (1.3).

In view of the ineuality (*),

$$d(Tx_0, Tx_n) \leq \emptyset(Tx_0) - \emptyset(Tx_n) \leq \emptyset(Tx_0).$$

Making $n \to \infty$ gives $d(Tx_0, fz) \le \phi(Tx_0)$. This ends the proof.

THEOREM 2. Let T, f be self-maps of X and $\phi: X \to [0, \infty)$. Suppose there exists an $x_0 \in X$ such that (CP) with A = X holds for every $fy \in O(T,f;x_0)$, and X is $(T,f;x_0)$-orbitally complete. Then :

(2.1) $\lim Tx_n = \lim fx_n = p$ exists, $p \in X$;

(2.2) $d(Tx_n, p) \le \phi(Tx_n)$;

(2.3) if {T,f} is a weakly x_0-preorbitally continuous pair and T, f are $(T,f;x_0)$-orbitally continuous then $Tp = fp$.

(2.4) $d(Tx_n, Tx_0) \le \phi(Tx_0)$ and $d(p,Tx_0) \le \phi(Tx_0)$.

PROORF. In view of the proof of Theorem 1 we need show only (2.3). Since $fx_n \to p$, $Tx_n \to p$, the w.x_0-p. commutativity of T and f implies $\lim d(Tfx_n, fTx_n) = 0$. This together with the orbital continuity of T and f gives $d(Tp, fp) = 0$, i.e. $Tp = fp$.

REMARKS 1. Theorem 3 of Bollenbacher and Hicks [1] generalizing the Caristi-Kirk fixed point theorem is obtained from Theorem 1 when A = X and f = I.

2. If $T(A) \subset f(A)$ then for any $x \in A$ there exists a $y \in A$ such that $fy=Tx$. Therefore $T(A) \subset f(A)$ implies the existence of an orbit $O(T,f;x_o)$ for every $x_0 \in A$. If A = X and T = I then $O(I,f;x_0) = \{x_n = fx_{n+1}: n = 0,1,2,\dots\}$.

3. If, instead of f(A), T(A) is assumed $(T,f;x_0)$-orbitally complete in Theorem 1 then, as evident from the proof of Theorem 1, fz and Tz are just interchanged in (1.1) - (1.4).

In view of Remark 2, we have the following corollaries from Theorems 1-2.

COROLLARY 1. Let T, f be maps on A with values in X such that $T(A) \subset f(A)$, $f(A)$ a (T,f)-orbitally complete subspace of X and (CP) for everyy $y \in A$. Then for any $x_0 \in A$, there exists a sequence $\{x_n\}$ in A such that $fx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots,$ and the conclusions (1.1) - (1.4) are ture.

COROLLARY 2. Let X be a metric space and T, f self-maps of X such that $T(X) \subset f(X)$, X is (T, f)-orbitally complete and (CP) with $A = X$ holds for ever $y \in X$. Then, for any $x_0 \in X$, there exists a sequence $\{x_n\}$ in X such that $fx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots,$ and the conclusions (2.1) - (2.2), (2.4) and the following are true :

(2.3′) if $\{T, f\}$ is a weakly preorbitally commuting pair and T, f are (T,f)-orbitally continuous then $Tp=fp$.

Taking $T = I$ in Theorem 2 we have the following fixed point theorem.

COROLLARY 3. Let $f: X \to X$ and $\phi : X \to [0,\infty)$. Suppose there exists an orbit $O(I,f;x_0)$ for some $x_0 \in X$ such that

(P) $\quad d(fy, y) \le \emptyset(fy) - \emptyset(y)$

for ever $y \in O(I,f;x_0)$, and X is $(I,f;x_0)$-orbitally complete Then :

(2.1c) $\lim x_n = p$ exists, $p \in X$;

(2.2c) $d(x_n,p) \le \emptyset(x_n)$;

(2.3c) $fp = p$ if f is $(I,f;x_0)$-orbitally continuous, that is p is a fixed point of f if it is continuous at p;

(2.4c) $d(x_n, x_0) \le \emptyset(x_0)$ and $d(p,x_0) \le \phi(x_0)$.

REMARK 4. Corollary 3 is a variant of [1, Th. 3] and an improved version of a result of Park [6, Prop.9]. It is noted in [6,p. 149] that the condition (P) includes a number of contractive type conditions for maps on X. Examples (see below) show that the range of applicability of our results is wider than *inter alia* Goebel's coincidence theorem (Theorem 3 below). He proved it using the Banach contraction principle, while, as also noted in [1, Remark 2], our results include many generalizaitons of Banach's fixed point theorem.

THEOREM 3[4]. Let T,f be maps on A with values in X such that $T(A) \subset f(A)$ and

(3.1) $\quad d(Tx, Ty) \leq q\, d(fx, fy)$

for all $x, y \in A$ and some $q \in (0, 1)$. If f(A) is a complete subspace of X then T and f have a coincidence point in A.

3. EXAMPLES

EXAMPLE 1. This example illustrates Theorem 1 and shows that Theorem 3 is not applicable. Let $A = \{a,b,c\}$; $X = \{1,2,3\}$; $Ta = Tb = 1$, $Tc = 2$; $fa = 1$, $fb = 2$, $fc = 3$; $\emptyset(1) = 5/2$, $\emptyset(2) = 4$, $\emptyset(3) = 6$; and d a metric on X such that $d(1, 2) = d(1,3) = 3/2$, $d(2, 3) = 2$. Then evidently $T(A) \subset f(A)$, $d(fx, Tx) \leq \emptyset(fx) - \emptyset(Tx)$ for every x in A, and a is a coincidence point of T and f. However, Theorem 3 is not applicalbe as T and f do not satisfy (3.1), since $d(Ta, Tc) = d(fa, fc)$.

EXAMPLE 2. Let X be the set of non-negative rationals, d the absolute value metric on X and T, f; $X \rightarrow X$ such that

$$Tx = \begin{cases} 2x^2, & x \leq 4 \\ x, & x > 4 \end{cases} \quad \text{and } fx = \begin{cases} 4x, & x \leq 1 \\ 8x, & x > 1. \end{cases} \text{ Let}$$

$$\emptyset(x) = \begin{cases} 1 + x, & x \leq 4 \\ 2 + x, & x > 4. \end{cases}$$

Then $T(X) \not\subseteq f(x)$ and $d(fx, Tx) \leq \phi(fx) - \phi(TX)$ for every x in X. Moreover, for $x_0 = 1$,

$$x_n = 2^{-2^n+1}, \; n = 1, 2, 3, \ldots, \text{ and}$$

$O(T,f;x_0) = \{2^{-2^n+3} : n = 1, 2, 3, \ldots\}$ converges to 0.

Evidently, Theorem 1 applies. Theorem 2 also applies as it is easy to check that $\{T,f\}$ is a $w.x_0$-p.c. pair. It may be noted that T and f are not weakly commuting since $d(Tfx,fTx) > d(fx,Tx)$ for $\frac{1}{2} \leq x \leq 1$.

EXAMPLE 3. Let (X, d) be as in Example 2, $\emptyset(x)=x$, $T = I$ and

$$fx = \begin{cases} 5x, & x \leq 1 \\ 1 + x/2, & x > 1. \end{cases}$$

Since, at $y = 4$, $d(fy, y) = 1$ and $\emptyset(fy) - \emptyset(y) = -1$, the condition (P) of Corollary 3 does not hold for all $y \in X$. However, since

$$O(I,f;1/5) = \{fx_n : x_n = 1/5^{n+1}, \; n = 1,2 \ldots\}$$

and $\quad d(fx_n, x_n) = 5^{-n} - 5^{-n-1} = \emptyset(fx_n) - \emptyset(x_n)$,

the condition (P) is satisfied for the orbit $O(I,f;1/5)$. So Corollary 3 applies to f, and $O(I,f;1/5)$ converges to 0, a fixed point of f. Indeed, the conclusions (2.1c) - (2.4c) are true for $x_0 = 1/5$. For $x_0 = 4$ we can have the following two orbits :

$$O_1 := \{4/5^n : n = 1, 2, 3 \ldots\}$$

and $\quad O_2 := \{2^n + 2 : n = 1, 2, 3, \ldots\}$.

It is easy to check that (P) is satisfied for O_1 and not for O_2. We land no where with O_2 while O_1 converges to 0, a fixed point of f.

EXAMPLE 4. This example shows that the conditions of Theorem 1 do not guarantee the existence of a common fixed point of maps T and f on X (=A). Let $X = \{x: x \geq 1/2\}$; $Tx = 2x$, $fx = 3x - 1/2$ and $\phi(x) = x$. Then $d(fx, Tx) = x-1/2 = \phi(fx) - \phi(Tx)$ for every x in X, and

$$O(T,f;1) = \{fx_n = 1 + (2/3)^{n-1} : x_n = 1/2 + 2^{n-1}/3^n, \; n=1,2,\ldots\} \to 1.$$

Evidenty $T\tfrac{1}{2} = f\tfrac{1}{2} = 1$, and Theorem 1 applies to the maps of this example, and the coincidence point $\tfrac{1}{2}$ is not a common fixed point of T and f. Indeed T and f are fixed point free maps.

EXAMPLE 5. This example shows that the conditions of Theorem 2 do not guarantee the existence of a common fixed point of T and f. Let $X = \{x : x \geq 1\}$, $Tx = x + 7/x$, $fx = (7x + 15)/8$, d the absolute value metric on X and

$$\phi(x) = \begin{cases} 22/x, & x = 1 \\ x, & x \neq 1. \end{cases}$$

Then $T(X) = [8, \infty) \subset [11/4, \infty) = f(X)$, and for $x_0 = 1$,

$$O(T,f;x_0) = \{fx_n = 8 : x_n = 7, \; n = 1, 2, 3, \ldots\} \to 8.$$

Moreover, as $d(fx_n, Tx_n) = 0 = \phi(fx_n) - \phi(Tx_n)$, the (CP) of Theorem 2 holds for the orbit O(T,f;1). To check the weak 1-preorbital commutativity, we note that T and f are commuting at $x_n (=7)$, n = 1, 2, ... Thus Theorem 2 applies to the maps T and f, and 8 is a coincidence point of T and f. Evidently T and f have no common fixed points. Indeed, T is a fixed point free map. Note that T and f are not weakly commuting.

Recall that, as shown by Examples 4-5, the conditions of Theorems 1-2 are not sufficient to ensure the existence of a common fixed point of maps T, f on A = X. So we have the following :

QUESTIONS 1. Under what additional condition(s), maps T, f on A=X satisfying the conditions of Corollary 1/Theorem 1/Theorem 2 will have a common fixed point ?

2. Will corollary 2 yeild a common fixed point when T and f are continuous or commuting (possibly both) ?

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FLOW OF BLOOD THROUGH A LOCALLY CONSTRICTED TUBE - A MATHEMATICAL MODEL AND ITS STATISTICAL ANALYSIS

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ABSTRACT

A knowledge of the wall shear stress distribution is very important for locating the arterial diseases and finding out mean to cure or alternatively to reduce the drag or wall shearing stress in cases of severe stenosis. Therefore in this paper an attempt has been made to study the flow behaviour of the particulate suspension through a locally constricted tube and to obtain the wall shear stress distribution for different values of time, axial positions and number densities. A statistical analysis has been carried out in order to achieve the uncovering of the embedded relations between the various parameters in the study with the help of multiple regression and partial correlations.

Key words : Stenosis, Red Blood Cells, Shear Stress
Subject Classification : Blood Rheology 92019

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1. INTRODUCTION :

Interest in problem of mechanics of system with more than one phase has developed repidly in recent years. Frequently we are concerned with the motion of a liquid or gas, which contains a distribution of solid spherical particles. Such situation occur, for example, in the movement of dust-laden air, in problem offluidization, in environmental pollution in the process by which rain drops are formed by the coalescence of small droplets which might be considered as solid particles for the purpose of examining their movement prior to coalescence, combustion and more recently in blood flow studies.

Saffman [13] described the effect of dust by two parameters, namely the concerntration and relaxation time. These parameters measure the rate at which the velocity of a dust particle adjusts to change the fluid velocity and depends upon the size of individual particles. Michael and Miller [9], Liu [8] have studied the flow produced by the motion of an infinite plate in a dusty gas occupying semi-infinite spce above it. Rao [12], Nath [10], Verma and Mathur [14] have studied the flow of dusty viscous liquid in circular channels. Healy and Young [3] have obtained exact solution for the problem using Laplace transform technique. Gupta and Gupta [2] studied the flow of dusty gases through a channel with arbitrary time varying pressure gradient. Charya [1] studied the flow of dusty fluid through a constricted channel.

Recently, the attention of the researchers has also focussed on the study of flow behaviour of the suspension of particles through a locally constrticted tube in reference to an important disease termed as "stenosis" by medical scientists. Blood flowing through a stenotic vein develops greater resistance and drag on

the tube walls and at times it happens to be fatal. Saffman's [13] dusty fluid serves as a better model to describe the blood as a binary system. Nayfeh [11] studied the pulsatile flow of liquid containing small solid particles through a long narrow circular tube. A knowledge of the wall shear stress distribution is very important for locating the arterial diseases and finding out means to cure or alternatively to reduce the drag or wall shearing stress in cases of severe stenosios. Therefore, in this paper an attempt has been made to study the flow behaviour of the particulate suspensions through a locally constricted tube and obtained wall shear stress distribution, through the study under a mathematical model. The model has been used to construct synthetic data, on an electronic computer using the numerical inversion formula of Laplace transform. This data has been statistically studied to bring forward the implications of the model. More importantly, the statistical validation of the model could be done only with a reliable real-life data, which the authors very much missed to have had.

2. FORMULATION OF THE MATHEMATICAL MODEL :

In a circular tube, we consider the origin on the axis of the tube and the x-axis along the axis of the tube. The motion the dusty viscous, incompressible fluid is considered along the axis of the tube under the influence of the time (t) - dependent pressure gradient. The boundary conditions for the fluid are same as those for classical fluid and no particle boundary conditions are required. For the present, the initial conditions are at $t \leq 0$.

$$u = v = \frac{\partial u}{\partial t} = 0$$

The appropriate momentum equations in cylindrical polar coordinates are :

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \frac{KNo}{\rho}(v-u) \quad \ldots (1)$$

$$m\frac{\partial v}{\partial t} = K(u-v) \quad \ldots (2)$$

Where r is radial distance, u and v are velocities of fluid and dust particles respectively, K is the Stoke's resistance coefficient which, for spherical particles of radius r is $6\pi\mu r$. Also, ρ and u are the density and viscosity of the blood respectively, and No is the number density of red blood cells, p is the pressure. Radius of constricted region may be expressed as [11]:

$$R = a - \alpha_o\left(1+\cos\frac{\pi z}{z_o}\right) \quad \ldots (3)$$

where 'a' is the radius of non-constricted region, α_o is constant depending on lumen of tube, z_o is the axial distance and $2z_o$ is constricted length.

Introducing the following nondimensional quantities

$$r^* = \frac{r}{a},\ x^* = \frac{x}{a},\ p^* = \frac{pa^2}{\rho\nu^2},\ t^* = \frac{t\nu}{a^2},\ u^* = \frac{ua}{\nu},\ v^* = \frac{va}{\nu} \quad \ldots (4)$$

In the equation (1) and (2) (droping the stars) we get

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \left(-\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + B(v-u) \quad \ldots (5)$$

$$\frac{\partial v}{\partial t} = \frac{u-v}{t} \quad \ldots (6)$$

where $B = \frac{M}{\tau} = \frac{N_o K a^2}{\mu}$, $M = \frac{No\,m}{}$

$$\tau = \frac{v\,m}{ka^2} \,;$$

m being the mass of red blood cells, the radius

$$\beta = 1 - \alpha\left(1 + \cos\frac{\pi z}{z_o}\right) \quad \ldots (7)$$

where, $\beta = \frac{R}{a}$, $\alpha = \frac{\delta}{a}$

and the pressure gradient may be assumed in the form

$$\frac{\partial p}{\partial x} = C_1 F(t) = e^{-\lambda t} \text{ (say)} \quad \ldots (8)$$

where, C_1 is a constant.

Equation (6) may be solved for v, using the particle initial condition $v(r,0) = 0$, to give

$$v(r,t) = \frac{1}{\tau} e^{-t/\tau} u(r,\eta) e^{\eta/\tau} d \quad \ldots (9)$$

On substituting (8) and (9) in equation (5) we get

$$\frac{\partial u}{\partial t} = C_1 F(t) + \left(\frac{\partial^2 u}{\partial t^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + B\left[\frac{1}{\tau} e^{t/\tau} \int_0^t u(r,\eta) e^{\eta/\tau} d\eta - u\right] \quad \ldots (10)$$

Once the above fluid momentum, equation is solved for u with the help of equation (3) for radius varying with axial distance, v can be obtained from equation (9).

3. SOLUTION OF THE PROBLEM :

Applying the Laplace transform in equation (5), we get

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r} - \left(\frac{s^2 - BK_2}{s}\right)\bar{u} + c_1 \bar{F}(s) = 0 \qquad \ldots (11)$$

where $K_2 = \frac{1}{\tau}$

Solving the equation (11) we get complimentary function as follows.

$$\bar{u} = aJ_0(Qr) + BY_0(Qr) \qquad \ldots (12)$$

where $Q = \frac{(s^2 + Bs - BK_2)^{1/2}}{s}$

J_0 and Y_0 are modified Bessel's functions.

Variation of the parameters method has been used to evaluate second order differential equation. We have obtained the solution for $\bar{u}$, using equation (12) as assumed solutions.

$\bar{u} = AJ_0\ (Qr) + BY_0\ (Qr)$

where

$$A = \frac{CY_0 r^2}{Q(Y_0 J_0' - J_0 Y_0')}\, dr + a \qquad \ldots (13)$$

$$B = \frac{CJ_0 r^2}{Q(Y_0 J_0' - J_0 Y_0')}\, dr + b \qquad \ldots (14)$$

As limit $r \longrightarrow 0$, $\bar{u}\ (r,s)$ should be finite, and at $r = \beta$ no slip condition should be satisfied. This leads to:

$$\bar{u} = D_1 \; J_0 \; (Qr) \; Y_0(Qr) - u_0 \qquad \ldots (15)$$

where

$$D_1 = \frac{C}{Q^3} \; (r^2 - \frac{2}{Q^2}) + \frac{2C}{Q^5} \log \; (r^2 - \frac{2}{Q^2}) + a$$

Now the value of u (as limit $r \longrightarrow o, u(r,s)$ should be finite and at $r \longrightarrow \beta$ no slip condition should be satisfied), can be evaluated using Laplace inversion formula, as follows:

$$u(r,t) = \frac{1}{2\pi i} \int_{r-\infty}^{r+\infty} \frac{1}{(s+)} (D_1 J_0 \; (Qr) - Y_0 \; (Q_r) - \bar{u}_0) e^{st} ds \qquad \ldots (16)$$

Therefore,

$$u(r,t) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \Big[\frac{s^{5/2}}{(s^2 + Bs - BK_2)^{5/2}} \; (r^2(s^2 + Bs - BK_2) - 2s)$$

$$+ 2\log \frac{(r^2(s^2 + Bs - BK_2) - 2s)}{(s^2 + Bs - BK_2)} + \frac{\bar{u}_0}{s} (1 + \frac{1}{2}\log(B, \beta^2)$$

$$+ \frac{1}{2} \left(\frac{s}{B} - \frac{K_2}{s}\right) - \frac{1}{4} \left(\frac{s}{B} - \frac{K_2}{s} \right)^2 \left(\frac{1}{s+}\right)$$

$$\{1 - \frac{1}{2}\log(B.r^2) - \frac{1}{2} \; (\frac{s}{B} - \frac{K_2}{s}) + \frac{1}{4}(\frac{s}{B} - \frac{K_2}{s})^2 \; \}] e^{st} ds$$

Then u(r,t) has been solved numerically by ussing inversion formula of Laplace transformation with the help of computer program [6]. Similarly v(r,t) can be found from equation (9) along with u(r,t). The skin friction on unit area of the surface of cylinder due to fluid is given by

$$F = \mu \left[\frac{\partial u}{\partial r} \right]_{r = 1} \qquad \dots (17)$$

Once the value of v is obtained, the skin friction is also found out by equation (17), by putting the value of v.

Velocities of liquid and solid particles and the skin friction on the wall have been represented grahically with variable axial positions, time and number densities of the suspension.

From figure (1) it is clear that the velocity of the fluid decreases as the number density of the dust particles increases. Velocity of the fluid also decreases with the axial position from the throat and the time that elapses. Fluid velocity is always maximum on the central axix.

From figure (2) it is clear that the particle velocity decreases as the number density increases. Particle velocity also decreases with axial position from the throat and the time elapsed. Further, from figure (1) and (2), we infer that at a particular axial position the fluid velocity is greater than the particle velocity. Near the boundary, the dust particles exhibit sudden changing velocity and this tendency decreases with the number density.

From figure (3) we find that skin friction increases with number density and decrease with time and axial position from the throat. Thus, in order to have lower values of the skin friction for blood type suspension the number density should be decreased.

This is frequently done by the dilution process in medical sciences. The above inferences from the mathematical model are characterized in a much more comprehensive manner by the statistical analyses in the following section.

4. STATISTICAL ANALYSES :

For the data generated per the mathematical model as displayed in figures 1, 2 and 3, simple statistical analyses were carried out. Multiple regression of u, v and F each on the other relevent independent variables and various partial correlation coefficient ([4],[5], & [7]) were computed in order to bring out the dependence structures

First we take to 'u', i.e., the velocity of the fluid.

When $N = 4 \times 10^7$.

(a) Case 1 : $z = 0$; It is found tnat,

$$u = 5.113 - 0.129t - 5.208r. \quad \ldots (18)$$

Also, the partial correlations,

$$R_{rt.u} = -0.051 \;;\; R_{ut.r} = -0.994 \;;\; R_{ur.t} = -0.052 \ldots (19)$$

(b) Case 2 : $z = 2$: we find that,

$$u = 2.044 - 0.252t - 2.084r. \quad \ldots (20)$$

And,

$$R_{rt.u} = -0.216 \;;\; R_{ut.r} = -0.992 \;;\; R_{ur.t} = -0.218 \ldots (21)$$

When, $N = 5 \times 10^7$:

(a) Case 1 : $z = 0$: We find that,

$$u = 5.861 - 0.752t - 5.237r. \quad \ldots (22)$$

And $$Rrt.u = -0.126 \; ; \; Rut.r = -0.968 \; ; \; Rur.t = -0.131 \quad \ldots (23)$$

(b) Case 2 : $z = 2$: We find that,

$$u = 2.357 - 0.088t - 2.277r. \quad \ldots (24)$$

And

$$Rrt.u = -0.088 \; ; \; Rut.r = -0.995 \; ; \; Rur.t = -0.088 \quad \ldots (25)$$

So (18), (20), (23) and (25) each show the average mathematical way 'u' goes down with 't' and /or 'r' going up. Also, as per (19), (21), (24), &(25)'u' and 't' are significantly related (opposite way) while the other relations are insignificant.

Now we take to 'v', i.e., velocity of solid particles.

When, $N = 4 \times 10^7$:

(a) Case 1 : $z = 0$ we find that

$$v = 0.769 - 0.121t - 0.679r \quad \ldots (26)$$

And

$$Rrt.v = -0.114 \; ; \; Rvt.r = -0.944 \; ; \; Rvr.t = -0.121 \quad \ldots (27)$$

(b) Case 2 : $z = 2$: We find that,

$$v = 1.676 - 0.276t - 2.147r.$$

And,

$$Rrt.v = -0.022 \; ; \; Rvt.r = -0.711 \; ; \; Rvrr.t = -0.311 \quad \ldots (28)$$

When, $N = 5 \times 10^7$:

(a) Case 1 : $z = 0$: We find that,

$$v = 0.845 - 0.132t - 0.755r. \quad \ldots (29)$$

And,

$$Rrt.v = 0.143 \; ; \; Rvt.r = -0.963 \; ; \; Rvr.t = -0.149 \quad \ldots (30)$$

(b) Case 2 : $z = 2$: We find that,

$$v = 0.346 - 0.110t - 0.338r.$$

And,

$$Rrt.v = -0.643 \; ; \; Rvt.r = -0.996 \; ; \; Rvr.t = -0.646 \quad \ldots (31)$$

The structures of the relationship are quite similar to those 'u' except that the correlations between 'r' and 't' and 'r' and 'u' is relatively not that insignificant as in case of 'u'.

Now we take to 'F', i.e. the skin friction.

When, $N = 4 \times 10^7$: We find that,

$$F = 0.480 - 0.087t - 0.163z. \quad \ldots (32)$$

and,

$$Rzt.F = -0.370 \; ; \; RFt.z = -0.995 \; ; \; RFz.t = -0.372 \quad \ldots (33)$$

When, $N = 5 \times 10^7$: We found that,

$$F = 0.688 - 0.095t - 0.205z. \quad \ldots (34)$$

And,

$$Rzt.F = -0.187 \; ; \; RFt.z = -0.985 \; ; \; RFz.t = -0.189 \quad \ldots (35)$$

Here also like for the case of 'u' only 'F' and 't' are significantly related the other pairs, namely, 'z' and 't' and 'F' and 'z' are not so significantly related.

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LATTICE SUM OF ELECTRIC FIELD GRADIENTS IN ORTHORHOMBIC CRYSTAL

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ABSTRACT

Electric field gradient (FEG) is calculated at a nuclear site in the Simple Orthorhombic crystal. The method uses Euler Maclaurin (EM) Summation formula and makes Plane Wise summation in direct crystal space.

Key words : Electric field gradient, Lattice Sum Convergence, Simple Orthorhombic Crystal, Euler-Maclaurin formula, Double Series.

1. INTRODUCTION

In a previous paper (Verma et al. 1983) a new technique to sum the electric field gradient (EFG) over lattice points at a nuclear site in simple tetragonal crystal was introduced. The technique involves repeated use of Euler-Maclaurin (EM) formula to evaluate the infinite series and the same formula was used to estimate error terms from a related finite sum. The lattice sum is evaluated inthe direct crystal space keeping the numerical computations reasonably low, unlike most of the previous methods using Fourier transform in reciprocal space.

In the present paper we use the above method (Verma et al. 1983) to calculate the Electric Field Gradient (EFG) at the nuclear site in simple orthorhombic crystal.

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2. DETAILS OF CALCULATION

The lattice part of electric field gradient (EFG) eq_{latt} is defined as

$$eq_{latt} = \sum_{\lambda} \frac{ze}{4\pi e_o} \frac{3z_\lambda^2 - r_\lambda^2}{r_\lambda^5} - \frac{ze}{4\pi e_o v} \int \frac{3z^2 - r^2}{r^5} d^3r \quad ...(1)$$

where,

z = charge on each ion

e = electronic charge

e_o = permittivity of vacuum

z = z-co-ordinate of the ion at th lattice point

r = distance of lattice point from the origin situated at the nucleus of interest

V = volume of the unit cell

The integration refers to the contribution from an assumed uniform background of negative charges. Both the terms in equation (1) are conditionally convergent, hence same boundary shape at infinity is to be assumed in summation and integration.

We employ plane wise summation which leads to a rapidly convergent expression of the sum. This means the crystal is assumed to be slab shaped with faces perpendicular to z-direction. With this boundry shape the integration $\int \frac{3z^2-r^2}{r^5} d^3r$ in equation (1) for the assumed boundary shape comes out to be $(-8\pi/3)$ (de Wette 1961).

In an orthorhombic lattice the lattice points are written as (n_1a, n_2b, n_3c), where n_1, n_2, n_3 run through all integers.

The summation in equation (1) can be written as

$$S = \frac{ze}{4\pi e_o a^3} \sum_{n_1,n_2,n_3=-\infty}^{\infty} f(n_1,n_2n_3) \quad ... (2)$$

where,

$$f(n_1, n_2, n_3) = \frac{2n_3B^2 - n_1^2 - n_2^2 A^2}{(n_1^2 + n_2^2A^2 + n_3^2 B^2)^{5/2}},$$

$A = b/a$ and $B = c/a$

The summation over n_1 and n_2 are to be carried out before n_3 summation.

2.1 Contribution from base plane ($z = n_3c = 0$)

The Contribution to $[\frac{4\pi\epsilon_o a^3}{ze} S]$ from the base plane is

$$U_o = \sum_{n_1, n_2 = -\infty}^{\infty} g(n_1, n_2)$$

$$= -4 \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} g(n_1, n_2) + 2 \sum_{n_1=1} g(n_1, 0)$$

$$- \frac{2}{A^3} \sum_{n_2=1}^{\infty} g(0, n_2) \qquad \ldots (3)$$

where,

$$g(n_1, n_2) = \frac{1}{(n_1^2+n_2^2A^2)^{3/2}}$$

The Euler-Maclaurin (EM) Summation formula (Hildebrand 1956) is

$$\sum_{n=M} f(n) = \int_M^N f(x)dx + \frac{1}{2} \{f(N) + f(M) + \int_M^N P_1(x) \frac{df(x)}{dx} dx \qquad \ldots (4)$$

where $P_1(x) = x - 1/2$ for $0 < X \leq 1$ and is defined outside this region by $P_1(x+1) = P_1(x)$.

Applying the EM formula (4) to (3) we get,

$$U_o = \frac{4}{A} \sum_{n_1=1}^{\infty} \frac{1}{n_1^2} \quad \frac{2}{A^3} \sum_{n_1=1}^{\infty} \frac{1}{n_1^3} - 4 \int_0^{\infty} P_1(x) \sum_{n_1=1}^{\infty} R \, dx$$

in which $R = \frac{d}{dx} g(n_1,x)$

Thus $$U_o = -\frac{4}{A} \xi(2) - \frac{2}{A^3}\xi(3) + \epsilon_o(\infty) \qquad \ldots (5)$$

Where $\epsilon_o(\infty)$ contains all integrals involving $P_1(x)$ terms and $\xi(2) = \pi^2/6$, $\xi(3) = 1.20205690$ are the Riemann Zeta functions at 2 and 3 respectively.

To Estimate $\epsilon_o(\infty)$ we apply the above procedure to the finite sum

$$U_o(N) = -\sum_{n_1,n_2=-N}^{N} g(n_1,n_2)$$

$$= -4\sum_{n_1=1}^{N}\sum_{n_2=0}^{N} g(n_1,n_2) + 2\sum_{n_1=1}^{N} g(n_1,0)$$

$$-\frac{2}{A^3}\sum_{n_2=1}^{N} g(0,n_2)$$

and get,

$$U_o(N) = -4\sum_{n_1=1}^{N} G(n_1,N) - \frac{2}{A^3}\sum_{n_2=1}^{N} g(0,n_2)$$

$$-2\sum_{n_1=1}^{N} g(n_1,N) - 4\int_o^N P_1(x)\sum_{n_1=1}^{N} R\,dx$$

Hence,

$$\epsilon_o(N) = -4\int_o^N P_1(x)\sum_{n_1=1}^{N} R\,dx$$

$$= -4\sum_{n_1=1}^{N}\sum_{n_2=0}^{N} g(n_1,n_2) + 4\sum_{n_1=1}^{N} G(n_1,N)$$

$$+2\sum_{n_1=1}^{N} g(n_1,0) + 2\sum_{n_1=1}^{N} g(n_1,N) \qquad \ldots\ldots.(6)$$

where,

$$G(n_1, N) = \frac{N}{n_1^2 (n_1^2 + N^2 A^2)^{1/2}}$$

As N iucreases $| \epsilon_o (\infty) - \epsilon_o (N) |$ decreases. It was shown by Verma et al. (1993) that the absolute difference between ( ) and (N) for a similar expression reduces below 10 as N is taken greater than 80. Our expression has a similar nature and hence we expect that to the accuracy of present calculations (five significant digits) $| \epsilon_o (\infty) - \epsilon_o (80) |$ may be neglected. Hence we calculate (N) for N = 80 from equation (6) and substitute it for (oo) in equation (5) to get the contribution due to base plane.

2.2 Contribution from the plane $z = n_3 c$ $(n \neq 0)$

The contribution of the plane $Z=n_3c$ to $(4\pi \epsilon_0 a^3 S/Ze)$ is

$$U_k = \sum_{n_1, n_2=-\infty}^{\infty} f(n_1, n_2, n_3)$$

$$= 4 \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} f(n_1, n_2, n_3) - 2 \sum_{n_1=1}^{\infty} f(n_1, 0, n_3)$$

$$+ 2 \sum_{n_2=0}^{\infty} f(0, n_2, n_3) - \frac{2}{K^3} \quad \ldots (7)$$

where $K = n_3 B$

Using EM formula (4) on summation over n_2 we get,

$$U_k = \frac{4}{A} \sum_{n_1=1}^{\infty} \frac{2K^2}{(K^2+n_1^2)^2} - \frac{4}{A} \sum_{n_1=1}^{\infty} \frac{1}{(K^2+n_1^2)} + \frac{2}{AK^2}$$

$$+ 2 \int_0^{\infty} P_1(x) R_2 dx + 4 \int_0^{\infty} P_1(x) \sum_{n_1=1}^{\infty} R_1 dx$$

$$= \frac{4}{A} \sum_{n_1=0}^{\infty} \frac{2K^2}{(K^2+n_1^2)^2} - \frac{4}{A} \sum_{n_1=0}^{\infty} \frac{1}{(K^2+n_1^2)} - \frac{2}{AK^2}$$

$$+ 2\int_0^{\infty} P_1(x)\, R_2 dx + 4\int_0^{\infty} P_1(x) \sum_{n_1=1}^{\infty} R_1 dx \qquad \ldots (8)$$

where $R_1 = \frac{d}{dx} f(n_1,x,n_3)$ and $R_2 = \frac{d}{dx} f(0,x,n_3)$

The first two series of equation (8) can be summed exactly using Cauchy's Residue formula (Spiegel 1964).

$$\sum_{n=0}^{\infty} \frac{1}{(K^2+n^2)^2} = \frac{\pi}{4} \left[\frac{\coth(\pi K)}{K^3} \quad \frac{\operatorname{cosech}^2(\pi K)}{K^2}\right] + \frac{1}{2K^4}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{K^2+n^2} = \frac{1+\pi K \coth(\pi K)}{2K^2}$$

So that equation (8) becomes

$$U_K = \frac{2\pi^2}{A} \operatorname{cosech}^2(\pi K) + e_k(\infty) \qquad \ldots (9)$$

where $e_k(\infty)$ contains all integrals involving $P_1(x)$ terms.

Again to estimate $e_k(\infty)$ we repeat the above procedure with the finite sum

$$U_k(N) = \sum_{n_1,n_2=-N}^{N} f(n_1,n_2,n_3)$$

$$= 4\sum_{n_1=1}^{N} \sum_{n_2=0}^{N} f(n_1,n_2,n_3) - 2\sum_{n_1=1}^{N} f(n_1,0,n_3)$$

$$+ \sum_{n_2=0}^{N} f(0,n_2,n_3) - \frac{2}{K^3}$$

and get

$$e_K(N) = 2\int_0^{N} P_1(x)\, R_2 dx + 4\int_0^{N} P(x) \sum_{n_1=1}^{N} R_1 dx$$

$$= \sum_{n_1,n_2=-N}^{N} f(n_1,n_2,n_3) + H(n_1,n_3,N) \qquad \ldots (10)$$

where,

$$H(n_1,n_3,N) = -\sum_{n_1=1}^{\tilde{N}} \frac{12\,K^2 N}{(K^2+n_1^2)^2(K^2+n_1^2+N^2A^2)^{1/2}}$$

$$+\sum_{n_1=1}^{N} \frac{4K^2\,N^3\,A^2}{(K^2+n_1^2)\,(K^2+n_1^2+N^2A^2)^{3/2}}$$

$$+\sum_{n_1=1}^{N} \frac{4\,N}{(K^2+n_1^2)\,(K^2+n_1^2A^2)^{\frac{1}{2}}}$$

$$-\,2\sum_{n_1=1}^{N} f(n_1,n_3,N) - f(0,n_3.N)$$

$$-\,\frac{4\,N}{K^2(K^2+N^2A^2)^{\frac{1}{2}}}\quad\frac{2\,N^3\,A^2}{K^2(K^2+N^2A^2)^{3/2}}$$

We have set N=80, and evaluated $e_k(N)$ from equation (10). This is substituted for $e_k(\infty)$ in equation (9) thereby neglecting $|\, e(\infty)-e_k(N)\,|$ and obtaining U_k.

3. Results and discussion

We made calculations for eight orthorhombic structures with different ratios of a:b:c (listed in table-1). Sets of lattice parameters correspond to the lattices of orthorhombic elements B, Ga, P(black), O, S(yellow), Br, I and U (Darken Lawrence et al.1953).

The contributions from base plane and other planes were evaluated from the equations derived above. The contribution falls rapidly as n_3 increases and that is the speciality of this method. In all our claculations a maximum of six planes above the origin need to be considered. The contribution from the planes below the origin was equal to that from the planes above by symmetry. The contribution $(-8\pi/3 \frac{ze}{4\pi\epsilon_o a^3})$ was added to get the eq_{latt} in c-direction.

For each set of lattice parameters the calculations were repeated thrice by choosing the z axis along each of the three edges of the cell and hence to get the EFG's along the three direction. These directions were then labelled according to the convention $|V_{zz}| \geq |V_{yy}| \geq |V_{xx}|$. The sum $V_{xx} + V_{yy} + V_{zz}$ is found to be zero within the limits of accuracy of present calculations as is expected from the Laplace equation.

As the total computations involved is of low order and the sum is made in direct crystal space without any special regrouping of charges, the use of Euler Maclaurin summation formula difinitely has interesting application in lattice summation exercises.

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TABLE - 1

Lattice EFG in Orthorhombic Lattice Structures at Substitutional Sites in units of Ze/4 C_o

Element	Lattice Parameters of (in K_x units) the simple orthorhombic Lattice a	b	c	V_{xx}	V_{yy}	EFG V_{zz} = eq
B	8.93	10.13	17.86	0.0013986	0.0039938	0.0053920
Ga	4.5107	4.516	7.6448	0.022063	0.022318	-0.044399
P (black)	3.31	4.38	10.50	0.010841	0.10415	-0.11499
O	3.44	3.82	5.50	0.017592	0.056615	-0.074201
S (yellow)	10.48	12.92	24.55	0.00047180	0.0028624	-0.0033361
Br	6.67	8.72	4.48	-0.0098774	-0.027391	0.037269
I	7.2501	9.7711	4.7761	-0.0082761	-0.023389	0.031667
U	5.865	4.945	2.852	-0.061142	-0.095374	0.15652

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EULER MACLAURIN SUMMATION FORMULA FOR TWO DIMENSIONS

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ABSTRACT

We here develop Euler Maclaurin Summation Formula (EMSF) for two dimension case so that it can be used in summing over a set of lattice points in a plane. Other three versions of the formula are also got.

AMS(MOS) Subject classifications (1980): 65 B 15

Key words : Lattice sum, Discrete mathematics, Double series summation.

1. INTRODUCTION

The EMSF for one dimension [1] is

$$\sum_{r=1}^{n} f(r) = \int_{1}^{n} f(x)dx + \frac{1}{2}\{f(1) + f(n)\} + \int_{1}^{n} P_1(x)\, f'(x)\, dx$$

where $P_1(x)$ is Bernoulli's polynomial of order one. But no such result like Euler Maclaurin Summation Formula (EMSF) is available in problems of two and three dimensions. We here develop the formula for two dimensions and similar extension for three dimensions is easy.

2. FORMULATION OF THE PROBLEM

Here we find the value of

$$S = \sum_{r=1}^{m} \sum_{s=1}^{n} f(r,s)$$

we first fix r and let s vary from 1 to n, thus

$$S = \sum_{r=1}^{m} \int_{1}^{n} f(r,y)dy + \frac{1}{2}\sum_{r=1}^{m} f(r,1) + \frac{1}{2}\sum_{r=1}^{m} f(r,n)$$

$$+ \sum_{r=1}^{m} \int_{1}^{n} P_1(y) f_y(r,y)dy$$

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we apply EMSF to each term of the R.H.S. and after arranging, get

$$S = \int_1^m \int_1^n f(x,y,)dxdy + \frac{1}{4}\{f(1,1)+f(m,1)+f(1,n)+f(m,n)\}$$

$$+\frac{1}{2}\{\int_1^n f(1,y)dy + \int_1^n f(m,y)dy + \int_1^m + f(x,1)dx + \int_1^m f(x,n)dx\}$$

$$+\frac{1}{2}[\int_1^n P_1(y)f_y(1,y)dy + \int_1^m P_1(x)f_x(x,1)dx +$$

$$+\int_1^n P_1(y)f_y(m,y)dy + \int_1^m P_1(x)f_x(x,1)dx]$$

$$+\int_1^m\int_1^n \{P_1(x)f_x(x,y) + P_1(y)f_y(x,y)\}dxdy$$

$$+\int_1^m\int_1^n P_1^P(x)P_1(y)f_{xy}(x,y)dxdy \qquad \ldots (1)$$

provided the partial derivatives used exist and are continuous.

In Green's theorem

$$\iint_D (Q_x - P_y)dxdy = \int_C (P\,dx + Q\,dy)$$

let C be the rectangle with vertices (1,1), (m,1), (m,n), (1,n) let D be the part of the plane enclosed by C and let $Q = -f(x,y) = -P$. then

$$\iint_D \{f_x(x,y)+f_y(x,y)\}\,dxdy = \int_C f(x,y)\,(dy-dx)$$

$$= -\int_1^m f(x,1)dx + \int_1^n f(m,y)dy + \int_1^m f(x,n)dx - \int_1^n f(1,y)dy \qquad \ldots (2)$$

Next we put $P = -P_1(x)f_x(x,y)$ and $Q = P_1(y)f_y(x,y)$, then

$$\int_1^m \int_1^n \{P_1(y)f_{xy}(x,y)+P_1(x)f_{xy}(x,y)\}dxdy$$

$$= -\int_1^m P_1(x)f_x(x,1)dx + \int_1^n P_1(y)f_y(m,y)dy + \int_1^m P_1(x)f_x(x,n)dx$$

$$- \int_1^n P_1(y)f_y(1,y)dy. \quad \ldots (3)$$

Thus (1) becomes, after putting values from (2) and (3)

$$S = \int_1^m \int_1^n \{f(x,y) - \frac{1}{2} f_x(x,y) - \frac{1}{2} f_y(x,y)\}dxdy$$

$$+ \frac{1}{4} \{f(1,1) + f(m,1) + f(1,n) + f(m,n)\}$$

$$+ \int_1^m \{f(x,n)\, dx + \int_1^n f(m,y)\, dy$$

$$+ \int_1^m P_1(x)\, f_x(x,n)\, dx + \int_1^n P_1(y)f_y(m,y)\, dy$$

$$+ \int_1^m \int_1^n \{P_1(x)f_x(x,y) + P_1(y)f_y(x,y)\}\, dxdy$$

$$+ \int_1^m \int_1^n [P_1(x)\, P_1(y) - \frac{1}{2} P_1(x) - \frac{1}{2}P_1(y)]f_{xy}(x,y)dx\, dy$$

$$\ldots (4)$$

This is our formula in most general case. We now try to discuss EMSF in two dimension for skew-symmetric and symmetric functions. We know that every function can be expressed as a sum of a symmetric and skew-symmetric function. Hence let

$$f(x,y) = g\,(x,y) + h(x,y),$$

$g(x,y)$ being symmetric and $h(x,y)$ skew symmetric.

The terms of

$$\sum_{r,s=1}^{n} h(r,s)$$

can be grouped in pairs of type $h(r,s) + h(s,r)$ and we are left with terms of the type $h(r,r)$, each of which is zero by skew symmetric property. Thus

$$\sum_{r,s=1}^{n} f(r,s) = \sum_{r,s=1}^{n} g(r,s)$$

Now g is symmetric i.e. $g(x,y) = g(y,x)$ Differentiating, $g_x(x,y) = g_x\,(y,x)$. Using these results and putting m=n , (1) reduces to

$$\sum_{r,s=1}^{n} f(r,s) = \sum_{r,s=1}^{n} g(r,s)$$

$$= \int_1^n\int_1^n g(x,y)\,dxdy + \frac{1}{4}\,\{g(1,1) + 2g(n,1)$$

$$+ g(n,n)\} + \{\int_1^n g(x,1)\,dx + \int_1^n g(x,n)\,dx\}$$

$$+ \{\int_1^n P_1(x)\, g_x(x,1)\,dx$$

$$+ \int_1^n P_1(x)\, g_x(x,n)\,dx\}$$

$$+ 2\int_1^n \int_1^n P_1(x)\, g_x(x,y)\,dxdy$$

$$+ \int_1^n \int_1^n P_1(x)\, P_1(y)\, g_{xy}(x,y)\,dxdy \qquad (5)$$

The last result can be written in terms of 3rd order Bernoulli polynomials if we integrate by parts twice the terms containing $P_1(x)$, and note that $P_3(x)$ is zero at integral arguments. So

$$\int_1^n P_1(x)\,\{g_x(x,1) + g_x(x,n)\}\,dx$$

$$= \frac{B_2}{2!}\,\{g_x(n,n) - g_x(1,1)\} + \int_1^n P_3(x)\,[g_{xxx}(x,1)$$

$$+ g_{xxx}(x,n)]\,dx \qquad \dots (6)$$

$$\int_1^n \int_1^n P_1(x)\, g_x(x,y)\,dxdy$$

$$= \frac{B_2}{2!}\int_1^n \{g_x(n,y) - g_x(1,y)\}\,dy$$

$$+ \int_1^n \int_1^n P_3(x) g_{xxx}(x,y)\, dxdy \qquad \dots (7)$$

$$\int_1^n \int_1^n P_1(x)\, P_1(y)\, g_{xy}(x,y)\, dxdy$$

$$= \frac{B_2}{2\,!} \{g_{xy}(n,n) - 2g_{xy}(n,1) + g_{xy}\ (1,1)\}$$

$$+ B_2 \int_1^n P_3(x)\, \{g_{xxxy}(x,n) - g_{xxxy}(x,1)\}\, dx$$

$$+ \int_1^n \int_1^n P_3(x)\, P_3(y)\, g_{xxxyyy}(x,y)\, dxdy \qquad \dots (8)$$

provided the partial derivatives used exist and are continuous so that change of order of differentiation is permitted.

Putting in (5) the above values as obtained in (6), (7) and (8) and arranging, we have,

$$\sum_{r,s=1}^{n} g(r,s)$$

$$= \int_1^n \int_1^n g(x,y)dxdy + \frac{1}{4} \{g(1,1) + 2g(n,1) + g(n,n)\}$$

$$+ [\ \frac{B_2}{2!} \{g_x(n,n) - g_x(1,1)\}$$

$$(\frac{B_2}{2!})^2 \ \{g_{xy}(n,n) - 2g_{xy}(n,1) + g_{xy}(1.1)\}]$$

$$+ \int_1^n \{g(x,1) + g(x,n) \ \ dx$$

$$+ B_2 \int_1^n \{g_x(n,y) - g_x(1,y)\}\, dy$$

$$+ \int_1^n P_3(x)\{g_{xxx}(x,1) + g_{xxx}(x,n)\}\, dx$$

$$+ B_2 \int_1^n P_3(x)\, \{g_{xxxy}(x,n) - g_{xxxy}(x,1)\}\, dx$$

$$+ 2 \int_1^n \int_1^n P_3(x) g_{xxx}(x,y)\, dx dy$$

$$+ \int_1^n \int_1^n P_3(x) P_3(y) g_{xxxyyy}(x,y)\, dx dy \qquad \ldots (9)$$

3. APPLICATION

A number of physical quantities are expressed as a negative powers of distance. Matter is a collection of discrete atoms. In rough calculation, matter is taken to be continuous and the physical quantities are calculated using double or triple integrals. This is done because much dificult mathematics is required if we consider the atoms at the positions where they are actually situated. Some people attack such a situation by a number transformations. If one wants to avoid transformations and avoid the assumption of continuous matter then one way of handling the situation is through the theorem developed in this paper. Here we give one such application using fourth power of distance.

I. J. Zukker and M.M. Robertson [2] calculated

$$\sum_{(m,n)\neq(0,0)}^{\infty} \frac{1}{(m^2+n^2)^2} = 6.0267002 \pm .00207895$$

using a number of Transformations and L functions. We, in order to calculate the above sum, take

$$\sum_{(m,n)\neq(0,0)}^{\infty} \frac{1}{(m^2+n^2)^2} = 4\zeta(4)+4\sum_{m,n=1}^{\infty} \frac{1}{(m^2+n^2)^2}$$

$$= 4\zeta(4) + 4\sum_{m,n=1}^{5} \frac{1}{(m^2+n^2)^2}$$

$$+ 8\sum_{n=1}^{5}\sum_{m=6}^{\infty} \frac{1}{(m^2+n^2)^2} + 4\sum_{m,n=6}^{\infty} \frac{1}{(m^2+n^2)^2}$$

The second term is a finite sum, the third has been calculated using EMSF for one dimension and the last one using our theorem for two dimensions. The result got is

$$6.0265284 \pm 0.005736$$

which compares well with [2].

4. REMARKS

The formula has a wide potentiality of application as most physical quantities are expressible as a point function is two or three dimensions. In many cases, form (2) is more suitable for numerical calculations than form (1).

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$$\sum_{(m,n)\neq(0,0)}^{\infty} (am^2 + bnm + cn^2)^{-s}$$

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ON SOME SELF-SUPERPOSABLE FLUID MOTIONS IN PROLATE SPHEROIDAL DUCTS

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ABSTRACT

In the present paper an attempt has been made to find some self-superposable motions of incompressible fluid in prolate spheroidal ducts. No stress has been given on boundary conditions and the solutions thus determined contain a set of constants. Pressure distribution of some of such flows has also been discussed. The curves along which the vorticity of the flow becomes constant have also been attempted. Some irrotational flows and the surfaces along which the flows show the tendency of irrotationality has also been determined. The aim of the paper is to introduce a method for solving the basic equations of fluid dynamics in prolate spheroidal coordinates by using the property of self-superposability.

1. INTRODUCTION

It was shown by Ballabh [2] that when a flow with velocity q satisfies the condition

$$\text{Curl}\ (\bar{q} \times \text{curl}\ \bar{q}) = 0 \qquad \ldots (1)$$

it becomes self-superposable. In the present paper the authors have attempted some fluid velocities for which $(\bar{q} \times \text{Curl}\ \bar{q}) = \bar{p}$ (say) can be represented as the gradient of a scalar quantity. Attention has been focussed on incompressible fluids only. For such fluids some velocities have been determined which satisfy the condition in prolate spheroidal system of coordinates and

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for which $\bar{p}$ can be represented as the gradient of a scalar quantty θ (say). The solutions thus found must be of self-superposable type. Each will have a set of constants which can be determined by the boundary conditions.

It has further been shown by Ballabh (3) that if $\bar{q}_1$ and $\bar{q}_2$ are self-superposable flows and $\bar{q}_1 \pm \bar{q}_2$ are also self-superposalbe then $\bar{q}_1$ and $\bar{q}_2$ are mutually superposable. By using this property some more self-superposable flows have been determined. Attempts have also been made to find some curves along which the vorticity of the flow becomes constant and also the conditions of irrotationality.

In this paper a method will be introduced to solve the basic equations of fluid dynamics in prolate spheroidal coordinates by using the property of self superposability. Mittal [5,6] Mittal et al.[7,8,9,10] have recently solved the equations for different coordinate systems.

FORMULATION OF THE PROBLEM

Let $\bar{q} \times \text{Curl } \bar{q} = \bar{p}$ ...(2)

For incompressible fluids, we have

$$\text{div } \bar{q} = 0 \quad \text{... (3)}$$

Now if a solution of equation (3) be found in such a way that p can be represented as the gradient of a scalar quantity say θ, i.e.,

$$\bar{p} \ \text{grad } \theta, \quad \text{... (4)}$$

it will give a self-superposable flow. It has been shown by Agarwal [1] that such solutions will also satisfy the equation of motion for a steady flow. For determining a flow of liquid let us consider the flow in prolate spheroidal coordinates (u, v, w). If q_u, q_v, q_w be the

components of q at any point (u, v, w) in prolate spheroidal coordinates [11] then in order to make equation (3) integrable we may consider the following cases :

Case I

Let $q_u = 0$. In this case equation (3) will be satisfied by a solution.

$$q_u = 0$$

$$q_v = \frac{AU(u)\ W(w)}{\sinh u .\ \sin v \sqrt{(\sinh^2 u + \sin^2 v)}} \quad \ldots (15)$$

$$q_w = \frac{BU_1\ (u)\ V(v)}{(\sinh^2 u + \sin^2 v)}$$

where U(u), $U_1(u)$ are integrable functions of u, V(v) and W(w) the integrable functions of v and w respectively, and A, B the constants.

For this fluid velocity, it can be shown that $\bar{p}$ can be represented by the gradient of a scalar quantity θ given by

$$\theta = \frac{A^2W^2}{a \sin^2 v} \int \frac{U\ U'\ du}{\sinh^2 u\ (\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A^2\ W^2}{a \sin^2 v} \int \frac{U^2 \text{ cosech } u .\ \text{Coth } u}{(\sinh^2 u + \sin^2 v)^{3/2}}\, du$$

$$+ \frac{B^2 U_1^2}{a \sinh u} \int \frac{V\ V' \quad dv}{\sin v\ (\sinh^2 u + \sin^2 v)^2}$$

$$+ \frac{B^2\ U_1^2}{a \sinh u} \int \frac{V^2\ (\sinh^2 u - \sin^2 v) \cos v}{(\sinh^2 u + \sin^2 v)^3 \sin^2 v}\, dv$$

$$+ \frac{A\ B\ U\ U_1}{a(\sinh^2 u+\sin^2 v)^4} \quad [\{V' + V \operatorname{Cot} v\} (\sinh^2 u+\sin^2 v) + V \sin 2v] \int W\, dw$$

$$- \frac{A^2\ U^2}{a(\sinh^2 u+\sin^2 v)^2 \sinh^2 u \sin^2 v} \int W\ W'\ dw \qquad \ldots (6)$$

here U(u), U_1(u), V(v), W(w) are represented by U, U_1, V and W respectively and U', U_1', V' and W' represent their differentials.

By choosing different suitable sets of values of U, U_1, V and W we may get a number of self-superposable fluid velocities. One of such velocities can be obtained by taking

$$U_1 = U = a \sinh u,\ V = a \sin v,\ W = a \cos w$$

$$\text{and } B = A \qquad \ldots. \quad (7)$$

thé fluid velocity will become ;

$$q_u = 0$$

$$q_v = \frac{A \operatorname{Cos} w}{\sin v \sqrt{(\sinh^2 u+\sin^2 v)}} \qquad \ldots. \quad (8)$$

$$q_w = \frac{A \sinh u.\ \sin v}{(\sinh^2 u + \sin^2 v)}$$

and ;

$$\theta = \frac{A^2}{a} \left[\frac{\operatorname{Cos}^2 w}{\sin^4 v} \left\{ \frac{1}{2 \sin v} \log \left(\frac{\sqrt{\sinh^2+\sin^2 v}-\sin v}{\sqrt{\sinh^2 u+\sin^2 v}+\sin v} \right) - \frac{1+\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}} \right\} + 2\sinh u \left\{ \frac{\sqrt{\sinh^2 u+\sin^2 v}+\sinh^2 u}{\sqrt{\sinh^2 u - \sin^2 v}} \right\}^{1/2}\right.$$

$$+ \frac{\sinh^2 u \operatorname{Cos} v \sin w(2 \sinh^2 u + 3 \sin^2 v)}{(\sinh^2 u + \sin^2 v)}$$

$$- \frac{.\ \sin^2 w}{2(\sinh^2 u + \sin^2 v)^2} \qquad \ldots (9)$$

If U, U_1, V, W are constants then

$$q_u = 0$$

$$q_v = \frac{C_1}{\sinh u.\ \sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (10)$$

$$q_w = \frac{D_1}{(\sinh^2 u + \sin^2 v)}$$

and

$$\theta = \frac{D_1^2}{a \sin^6 v} \left[\frac{3 \sin^{-1}}{2 \sin v} \left(\frac{\sin v}{\sqrt{\sin^2 u + \sin^2 v}} \right) - \frac{\sinh u}{2(\sinh^2 u + \sin^2 v)} - \frac{1}{\sinh u} \right]$$

$$- \frac{2D_1^2 \sinh u}{a \sin^3 v \sqrt{(\sinh^2 u + \sin^2 v)}} +$$

$$\frac{D_1^2}{a \sinh^6 u} \left[\frac{3}{2 \sinh u} \sin^{-1} \left(\frac{\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}} \right) \right.$$

$$\left. - \frac{\sin v}{2(\sinh^2 u + \sin^2 v)} - \frac{1}{\sin v} \right]$$

$$- \frac{2 D_1^2 \sin v}{a \sinh^3 u \sqrt{(\sinh^2 u + \sin^2 v)}}$$

$$+ \frac{C_1 D_1}{a} \left[\frac{\sinh u . \operatorname{Cot} v \, (\sin^2 v - \sinh^2 u)}{(\sinh^2 u + \sin^2 v)^3} \right] w \qquad \dots (11)$$

Case II

(i) When $q_v = 0$, In this case, the self-superposable flows may be

$$q_u = \frac{A_1 V_1 W_1}{\sinh u \sin v \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = 0$$

$$q_v = \frac{B_1 U_2 V_2}{(\sinh^2 u + \sin^2 v)} \qquad \dots (12)$$

and

$$\theta = - \frac{A_1 B_1 V_1 V_2 W_1'}{a \sin^3 v} \int \frac{U_2 \, du}{\sinh^3 u \, (\sinh u + \sin^2 v)}$$

$$+ \frac{B_1^2 V_2^2}{a \sin v} \int \frac{U_2^2 (\sin^2 v - \sinh^2 u) \operatorname{Cosh} u \, du}{\sinh^2 u \, (\sinh^2 u + \sin^2 v)^3}$$

$$+ \frac{B_1^2 V_2^2}{a \sin v} \int \frac{U_2' U_2 \, du}{\sinh u \, (\sinh^2 u + \sin^2 v)^2}$$

$$+ \frac{B_1^2 U_2^2}{a \sinh u} \int \frac{V_2^2 \operatorname{Cos} v \, (\sinh^2 - \sin^2 v)}{\sin^2 v \, (\sinh^2 u + \sin^2 v)^3} dv$$

$$+ \frac{B_1^2 U_2^2}{a \sinh u} \int \frac{V_2 V'_2 \, dv}{\sin v (\sinh^2 u + \sin^2 v)^2}$$

$$+ \frac{A_1^2 W_1^2}{a \sinh^2 u} \int \frac{(V_1 V_1' \operatorname{Cosec} v - V_1^2 \operatorname{cosec} v \cot v)}{\sin v(\sinh^2 u + \sin^2 v)} dv$$

$$+ \frac{A_1^2 V_1^2}{a \sinh^2 u \sin^2 v (\sinh^2 u + \sin^2 v)^2} \int W_1 W_1' \, dw$$

$$- \frac{U'_2 \sinh u}{(\sinh^2 u + \sin^2 v)}] \int W_1 \, dw \qquad \ldots. (13)$$

(ii)

$$q_u = \frac{A_1 \operatorname{Cos} w}{\sinh u \sqrt{(\sinh^2 u + \sin^2 v)}}$$

$$q_v = 0 \qquad \ldots (14)$$

$$q_w = \frac{A_1 \sinh u . \sin v}{(\sinh_u^2 + \sin^2 v)}$$

and

$$\theta = - \frac{A_1^2 \sin w}{a \sin^3 v} [\operatorname{Coth} u - \frac{2}{\sin^2 v} \tan^{-1} (\frac{\tanh u}{\tan v})]$$

$$+ \frac{A_1^2 \sin v}{a} [\frac{2 \sinh u}{(\sinh^2 u + \sin^2 v)} - \frac{(2 \sin^2 v + 1)}{\sin v}$$

$$\sin^{-1} (\frac{v}{\sqrt{\sinh^2 + \sin^2 v}}) - \frac{1}{4 \sin^3 v} \sin (\frac{2 \sin v}{\sqrt{\sinh^2 u + \sin^2 v}})$$

$$+ \frac{A_1^2}{a} \operatorname{Cosec} v - \log (\frac{\sinh^2 u}{\sqrt{\sinh^2 u + \sin^2 v}})$$

$$\sin^{-1}\left(\frac{u}{\sqrt{\mathrm{Sinh}^2+\sin^2 v}}\right) - \frac{1}{4\sinh^3 u}\sin\left(\frac{2\sinh u}{\sqrt{\sinh^2 u+\sin^2 v}}\right)$$

$$+\frac{A_1^2}{2a}\quad \mathrm{cosech}^2 u\left[\frac{\sinh u.\ \sin v}{(\sinh^2+\sin^2 v)} - \sin^{-1}\left(\frac{\sinh u}{\sqrt{\mathrm{Sinh}^2 u+\sin^2 v}}\right)\right]$$

$$-\frac{A_1^2}{2a}\quad \frac{\sin^2 w}{\sinh^2 u(\sinh^2 u+\sin^2 v)^2}$$

$$-\frac{2A_1^2}{a}\quad\frac{\sin^4 v\ \mathrm{Cosh}\ u\ \sin w}{(\sinh^2 h+\sin^2)^3} \qquad \dots\quad (15)$$

(iii)

$$q_u = \frac{C_2}{\sinh u\ \sin v\ \ \sinh^2 u+\sin^2 v}$$

$$q_v = 0$$

$$q_w = \frac{2}{(\sinh^2 u+\sin^2 v)} \qquad \dots\quad (16)$$

and

$$\theta = \frac{D_2^2}{a\sin v}\ [\mathrm{Cosech}^7 u\ \{\sin v.\ \mathrm{Cosech}\ u$$

$$-\frac{1}{32}\sin\left(4\sin^{-1}\left(\frac{\sin v}{\sqrt{\sinh^2 u+\sin^2 v}}\right)\right)$$

$$-\frac{\sin v.\sinh u}{(\sinh^2 u+\sin^2 v} - \frac{(9+8\sin^4 v)}{8}\sin^{-1}\left(\frac{\sin v}{\sqrt{\sinh^2 u+\sin^2 v}}\right)$$

$$-\frac{\sinh u}{\sin^2 v(\sinh^2 u+\sin^2 v)}\ ]$$

$$+\frac{D_2^2}{a\sinh u}\ [\mathrm{Cosech}^7 u\ \{\sinh u.\ \mathrm{Cosec}\ v$$

$$- \frac{1}{32} \sin \left(4 \sin^{-1} \left(\frac{\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}}\right)\right)$$

$$- \frac{\sin v. \sinh u}{(\sinh^2 u+\sin^2 v)} - \frac{(9+8 \sinh^2 u)}{8} \sin^{-1}\left(\frac{\sinh u}{\sqrt{\sinh^2 u + \sin^2 v}}\right)$$

$$- \frac{\sin v}{\sinh^2 u\ (\sinh^2 u + \sin^2 v)}]$$

$$+ \frac{C_2 D_2 \sin v.\coth u\ (\sin^2 v-\sinh^2 u)\, w}{(\sinh^2 u + \sin^2 v)} \quad .... (17)$$

Case III :- When $q_w = 0$, some self-superposable flow may be:

(i) $$q_u = \frac{A_2\ V_3\ W_2}{\sinh u.\sin v \sqrt{\sinh^2 u+\sin^2 v}}$$

$$q_v = \frac{B_2\ U_3\ W_3}{\sinh u.\sin v \sqrt{\sinh^2 i+\sin^2 v}}$$

$$q_w = 0 \quad ... (18)$$

and

$$\theta = \frac{B_2^2\ W_3^2}{a \sin^2 v} \int \frac{(U_3'\ U_3 \sinh u- U_3^2 \operatorname{Cosh} u)\,du}{\sinh^3 u\ (\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A_2 B_2 W_3 W_2}{a \sin^3 u} \{V_3' \sin v - V_3 \operatorname{Cos} v\} \int \frac{U_3 \operatorname{Cosech}^2 u \quad du}{(\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A_2 B_2 W_3 W_2}{a \sin^3 v} \{V_3' \sin v - V_3 \operatorname{Cos} v\} \int \frac{U_3 \operatorname{Cosech}^2 u \quad du}{(\sinh^2 u + \sin^2 v)^{3/2}}$$

$$- \frac{A_2 B_2 W_3 W_3}{a \sin^3 v} \{U_3' \sinh u - U_3 \operatorname{Cosh} u\} \int \frac{V_3 \operatorname{Cosec}^2 v \quad dv}{(\sinh^2 u + \sin^2 v)^{3/2}}$$

$$+ \frac{A_2^2 W_2^2}{a \sinh^2 u} \int \frac{(V_3' V_3 \sin v - V_3^2 \operatorname{Cos} v)}{\sin^3 v (\sinh^2 u + \sin^2 v)^{3/2}} dv$$

$$+ \frac{A_2^2 V_3^2}{a \sinh^2 \sin^2 v (\sinh^2 u + \sin^2 v)} \int W_2 W_2' dw$$

$$+ \frac{B_2^2 U_3^2}{a \sinh^2 u \sin^2 v (\sinh^2 u + \sin^2 v)} \int W_3 W_3' dw \qquad \ldots (19)$$

(ii)

$$q_u = \frac{A_2 \operatorname{Cos} w}{\sinh u \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = \frac{A_2 \operatorname{Cos} w}{\sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (20)$$

$$q_w = 0$$

and

$$\theta = \frac{A_2^2}{2a} \operatorname{Cosech}^2 u . \operatorname{cosec}^2 v . \cos^2 w \qquad \ldots. (21)$$

(iii)

$$q_u = \frac{C_3}{\sinh u \ \sinh^2 u + \sin^2 v)} \qquad \ldots (22)$$

$$q_v = \frac{D_3}{\sin v (\sinh^2 u + \sin^2 v)}$$

$$q_w = 0$$

$$\theta = \text{Constt.} \qquad \ldots (23)$$

Case IV

When $q_u = q_v = 0$ a possible solution of equation (3) is given by

$$q_u = 0$$

$$q_u = 0$$

$$q_w = \frac{A_3 \, U_u \, V_u}{(\sinh^2 u + \sin^2 v)} \qquad \ldots (24)$$

Case V

When $q_v = 0$, $q_w = 0$, the self superposable flow is given by

$$q_u = \frac{A_4 \, V_5 \, W_4}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_v = 0 \qquad \ldots (25)$$

$$q_w = 0$$

Case VI

When $q_w = 0$, $q_u = 0$, the flow is

$$q_u = 0$$

$$q_v = \frac{A_5 \, U_5 \, U_5}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \ldots (26)$$

$$q_w = 0$$

In all the above cases U_n, V_n, W_n ($n = 1, 2, 3, 4 \ldots\ldots$) are integrable functions of u, v and w respectively and A_n, B_n, C_n, D_n ($n=1, 2, 3, \ldots.$) are constants whch may be determined by boundary conditions.

SUPERPOSABLE FLUID MOTION

It has already been shown that the hydrodynamic flows given by equations (5) and (25) are self-superposable. It can also easily be shown that if, $\bar{q}_1$ and $\bar{q}_2$ be the two flows given by equation (5) and (25), then $\bar{p}$ for $\bar{q}_1 \pm \bar{q}_2$ can also be represented by the gradient of scalar quantities. Thus $\bar{q}_1$ and $\bar{q}_2$ will be mutually superposable and a flow

$$q_u = \frac{A\ V(v)\ W(w)}{\sinh u . \sin v \sqrt{\sinh^2 u + \sin^2 v}} \qquad \dots (27)$$

$$q_v = \frac{B\ U(u)\ W(w)}{\sinh u.\ \sin v \sqrt{\sinh^2 u + \sin^2 v}}$$

$$q_w = \frac{C\ U(u).\ V(v)}{(\sinh^2 u + \sin^2 v)}$$

is possible. The same same flow can be determined by mutually superposing the flows (12) and (26) and (18) and (24).

PRESSURE DISTRIBUTION

It is interesting to note that 0 is nothing but Bernoulli function given by (4)

$$\theta = \left(\frac{q^2}{2g}\right) + h + \frac{p}{g} \qquad \dots (28)$$

where q, g, h and p denote velocity, acceleration due to gravity, height above some hoprizontal plane of reference and the pressure head.

It is a well known fact that for an incompressible fluid the pressure head p is given by [4]

$$p = \frac{P}{\rho_o} + \text{constant.}$$

where P is the pressure distribution.

Also if the motion of the fluid be steady and slow then the value of h can be taken, without much loss of generality as u for the flows (5), (8) and (10); as v for the flow (12), (14) and (16); and w for (18), (20) and (22).

Thus for the flow (20) the pressure distribution

$$= K_1+K_2w + K_3\text{Cos}^2w \frac{(\text{Cosech}^2u + \text{Cose}^2v)}{(\sinh^2u + \sin^2v)}$$

$$+ K_4 \text{ Cosech}^2u \text{ cosec}^2v \cos^2w \qquad \ldots (29)$$

Similarly for the flow (22) taking $C_3 = D_3$

we have

$$P = K_5 + K_6w + K_7 \frac{(\text{cosech}^2u + \text{cosec}^2 v)}{(\sinh^2u + \sin^2v)} \qquad \ldots (30)$$

where K_1, K_2, K_3, K_4, K_5, K_6, K_7 are constants.

Similarly the pressure distribution for the other flows can be determined.

VORTICITY OF THE FLOW

It was shown by Ballabh [3] that for a self-superposable flow, verticity is constant along its stream lines. If $\bar{T}$ is a unit tangent along a stream line then

$$T \times q = 0 \qquad \ldots (31)$$

By equation (16) and (31) it can readily be shown that

$$T = \left[\frac{\sinh^2u + \sin^2v}{\{\sin^2v(\sinh^4 + \sinh^2u) + \sinh^4u + \sin^4v\}^{1/2}}, 0, \frac{\sinh^2 . \sin v}{\sin^2v(\sinh^4u+\sinh^2u) + \sinh^4u+\sin^4v\}^{1/2}}\right] \qquad \ldots (32)$$

Hence the vorticity of the flow (16) is constant along the curve represented by equation (32). Similarly the curves of constant velocity can also be found for other flows,

IRROTATIONALITY

Vorticity $\bar{G}$ for the flow (10) can be calculated

$$\bar{G} = \frac{D_1}{a \sin v(\sinh^2 u+\sin^2 v)^{1/2}} \left\{\frac{(\sinh^2 u-\sin^2 u) \operatorname{Cos} v}{(\sinh^2 u + \sin^2 v)^2}\right\} \bar{e}_1$$

$$- \frac{D_1}{a \sinh u(\sinh^2 u+\sin^2 v)^{1/2}} \left\{\frac{\cosh u\,(\sin^2 v-\sinh^2 u)}{(\sinh^2 u + \sin^2 v)^2}\right\} \bar{e}_2$$

$$-C_1 \frac{\operatorname{Cosech} u.\ \operatorname{Coth} u.\ \operatorname{Cosec} v}{a\,(\sinh^2 u+\sin^2 v)} \bar{e}_3 \qquad \ldots.\ (33)$$

It is clear from equation (33) that flow (10) is not irrotational. For the flow (22)

$$\bar{G} = 0.$$

Hence the flow (22) is irrotational throughout.

Similar conclusions can be drawn for the other flow discussed earlier.

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EFFECT OF NON-HOMOGENEITY OF CRITICAL LOAD IN BUCKLING OF A BAR UNDER DISTRIBUTED AXIAL LOAD

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ABSTRACT

Critical load for buckling of a bar of non-homogeneous material has been obtained by two methods : (i) by using the differential equation of the deflection curve and (ii) by the energy method. The non-homogeneity of the bar has been assumed to be arising out of the variation of Young's modulus of the bar with the longitudinal distance. Critical loads obtained by the methods are found to be nearly same. The effect of non-homogeneity of this type has been observed to increase the critical load of the associated homogeneous case.

1. INTRODUCTION

The beam problem is very well-known problem in the theory of elasticity and it has significant importance in engineering structures. Thus from structural point of view, knowledge of critical buckling load of such beams is essential. The critical loads for buckling for a beam of homogeneous material under different conditions have been obtained by many investigators (Timoshenko and Gere [2]). But if the material is non-homogeneous a complete solution of the governing differential equation may not always be

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possible and in such a case approximate methods e.g. energy method, other variational methods or purturbation method, are to be used to get approximate critical load for buckling.

Our present investigation aims at the effect of non-homogeneity on critical load for buckling of a bar whose Young's modulus varies Linearly with its longitudinal distance. The Origin of co-ordinates has been suitably chosen in order to simplify the governing differential equation of the deflection curve. Solving this differential equation and using necessary and conditions critical load for buckling has been obtained. Critical load for buckling has also been obtained by assuming an approximate shape of the deflection curve and then by using the energy method. As is expected, (Timoshenko [1]) the critical load found by the energy method is slightly grater than that found by the first method. Of course, by increasing the number of parameters in the approximate equation for the deflection curve the difference of two critical load values may be made sufficiently small. In this paper it has been observed that critical load in the non-homogeneous case is greater than that in the associated homogeneous case.

2. FORMULATION OF THE PROBLEM

Let us consider a bar of length l whose lower end is vertically built in, upper end is free and the weight is uniformly distributed along the length.

We choose the co-ordinate axes as shown in the figure. If the bar buckles as shown by the dotted line in the figure, the differential equation of the curve is

$$EI \frac{d^2y}{dx^2} = \int_x^b q\,(\eta - y)\, d\xi \qquad \ldots (1)$$

where $b = a + \ell$, $a = OA$ (in the figure), and the integral on the right hand side of (1) represents the bending moment of any cross section fg produced by the uniform distributed load of intensity q.

Differentiation of (1) with respect to x gives

$$\frac{d}{dx}\left[EI\,\frac{d^2y}{dx^2}\right] = -q\,(b-x)\,\frac{dy}{dx} \qquad \ldots (2)$$

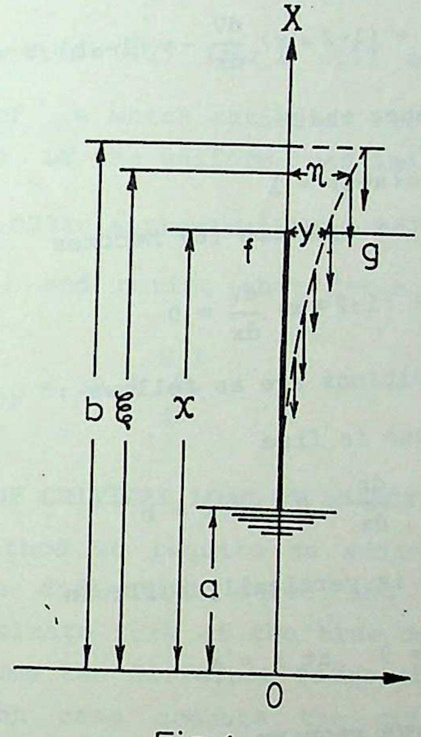

Fig.1

We now assume that the Young's modulus E of the bar obeys the law

$$E = E_o\left(\frac{x}{a}\right),\ a \le x \le b \qquad \ldots (3)$$

Where E_o is the value of E at the lower end of the bar and a is a suitable length to be defined latter.

Using (3) and writing $z = \frac{dy}{dx}$, equation (2) becomes where

$$x \frac{d^2 z}{dx^2} + \frac{dz}{dx} + \alpha^2 (b-x) z = 0 \qquad \ldots (4)$$

$$\alpha^2 = \frac{qa}{E_o I} \qquad \ldots (5)$$

Setting $z = e^{-\alpha x}.V$, equation (4) becomes

$$x \frac{d^2 V}{dx^2} + (1-2\alpha x) \frac{dV}{dx} - \alpha (1-\alpha b) V = 0 \qquad \ldots (6)$$

we now choose a such that

$$\alpha b = \alpha(a+\ell) = 1$$

with this choice the equation (6) becomes

$$x \frac{d^2 V}{dx^2} + (1-2\alpha x) \frac{dV}{dx} = 0 \qquad \ldots (8)$$

The boundary conditions are as follows :

Since the upper end is free

$$\frac{d^2 V}{dx^2} = \frac{dz}{dx} = 0 \quad \text{at } x = b$$

and the lower end is vertically built in

$$\frac{dy}{dx} = z = 0 \quad \text{at } x = a \qquad \ldots (9b)$$

3. SOLUTION OF THE PROBLEM

The solution of (8) is

$$V = A\,H(x) + B$$

Where $H(x) = \int \frac{e^{2\alpha x}}{x} dx$

and A and B are constants.

Thus $$z = e^{-\alpha x}.[A\,H(x) + B] \qquad \ldots (10)$$

Using boundary condition (9a) we get

$$z = A e^{-\alpha x}[H(x)-H(b)+e^2] \qquad \dots (11)$$

Where (7) has been used.

Condition 9(b) gives

$$\int_a^b \frac{e^{2\alpha x}}{x}\, dx = e^2$$

which by (7) becomes

$$\int_{2\alpha a}^{2} \frac{e^t}{t}\, dt = e^2 \qquad \dots (12)$$

The roots of equation (12) in αa will make $z=0$ at $x=a$ The lowest value of a which satisfies (12) corresponds to the critical value of the uniform load and is found to be

$$\alpha a = 0.0234 \text{ approximately.}$$

Using (7) and (5) and noting that $\ell = b-a$ we compute the critical load as

$$(q\ell)_{cr} = 39.8 \frac{E\, I_o}{\ell^2} \qquad \dots (13)$$

4. COMPUTATION OF CRITICAL LOAD BY ENERGY METHOD

In this method we require an advance knowledge of the shape of the deflection curve and so we have to consider an approximate form of the true deflection curve. Here we shall assume two different forms of the deflection curve and in each case compute the critical load for buckling. Since the critical load for true deflection curve is minimum, comparing the above two values of critical loads we can select the better of the two approximate curves.

Let the approximate deflection curve be $y = f(x)$ Here we consider the following two forms of $f(x)$.

4.1 f(x) is a Tribonometric Function of x

Here we approximate the true deflection curve by taking

$$y = \delta_1(1-\cos\frac{\pi}{2}x) + \delta_2(1-\mathrm{Cos}\frac{3\pi}{2}x) + \delta_3(1-\mathrm{Cos}\frac{5\pi}{2}x) \qquad \ldots (14)$$

where $\quad (b-a)\,X = x-a$ $\qquad \ldots (15)$

and δ_i (i = 1,2,3) are parameters.

The deflection y in (14) then satisfies the end conditions (9a) and (9b).

The bending moment at any cross section fg in the figure is

$$M = \int_x^b q(\eta - y)\,d\xi = q(b-a)\sum_{r=1}^{3} \delta_r S_r$$

Where $S_r = (1-x)\,\mathrm{Cos}\,(\theta_r X) + \frac{1}{\theta_r}\,[(-1)^r + \mathrm{Sin}(\theta_r X)]$

and $\quad \theta_r = \frac{\pi}{2}\,(2r-1)$

The strain energy of bending is given by [Timoshenko and Gere [2] (1961), p.105]

$$\Delta U = \int_a^b \frac{M^2}{2EI}\,dx = \frac{q^2(b-a)^3}{2\,E_oI}\sum_{r,s=1}^{3} I_{rs}\,\delta_r\,\delta_s$$

Where $I_{rs} = \int_o^1 S_r S_s\,F(x)\,dx$ $\qquad \ldots (16)$

and $\quad F(x) = 1/[1+(b/a - 1)\,x]$

In view of (14) the total work produced by the load during buckling is, [Timoshenko and Gere [2] (1961), p.108]

$$\Delta T = \frac{1}{2}\,q\int_a^b (b-x)\,(\frac{dy}{dx})^2\,dx = \frac{\pi^2}{8}\,q\sum_{r,s=1}^{3} \beta_{rs}\,\delta_r\,\delta_s \,.. \quad (17)$$

where $\beta_{rr} = \frac{r^2}{4} - \frac{1}{\pi^2}$, r = 1,2,3

$$\beta_{12} = \frac{3}{\pi^2}\ ,\ \beta_{13} = -\frac{5}{\pi^2}\ ,\ \beta_{23} = \frac{15}{\pi^2}\ ,\ \beta_{rs} = \beta_{sr}$$

The critical value of the axial load satisfies the equation, [Timoshenko and Gere [2] (1961), p.84]

$$\Delta U = \Delta T \qquad \ldots (18)$$

Substituting (16) and (17) in (18) we get

$$(q)_{cr} = P\left(\sum_{r,s=1}^{3} \beta_{rs}\,\delta_r\,\delta_s\right) \Big/ \sum_{r,s=1}^{3} I_{rs}\,\delta_r\,\delta_s \qquad \ldots (19)$$

where $P = \dfrac{\pi^2 E_o I}{4\ell^3}$

We now choose parameters δ_r $(r=1,2,3)$ such that $(q)_{cr}$ becomes minimum. The conditions for that are

$$\frac{\partial}{\partial \delta_r}(q)_{cr} = 0,\ r=1,2,3. \qquad \ldots (20)$$

From (19) and (20) and from the fact that the possibility of buckling occurs when δ_r $(r=1,2,3) \neq 0$, we get

$$\det(a_{rs}) = 0,\ r,\ s = 1,2,3 \qquad \ldots (21)$$

where

$$a_{rs} = \beta_{rs} - \frac{(q)_{cr}}{P} I_{rs},\ r,s = 1,2,3. \qquad \ldots (22)$$

Now using (7) we find that

$$\frac{(q)_{cr}}{P} \quad \frac{4(u-1)^3}{\pi^2 u^2} = h\ (u) \qquad \ldots (23)$$

where $u = \dfrac{b}{a}$

Substituting (23) in (22) we get a_{rs} as functions of b/a. Equation (21) then gives the value of b/a corresponding to the critical load.

Solving (21) we obtain

$$\frac{a}{b} = 0.0228 \text{ approximately,}$$

Using (7) and noting that $\ell = b-a$ we compute the critical load as

$$(q\ell)_{cr} = 40.92 \frac{E_oI}{\ell^2} \qquad \ldots (24)$$

Comparing (13) with (24) we find that with the approximate deflection curve given by (14) the error in the value of the critical load by the energy method is less than 2.8 percent. It may be mentioned here that if we put $\delta_3=0$ in (14) or in otherwise if there are only two parameters in the deflection curve the error in the critical load was observed to be 7.26 percent. Instead of (14) had we assumed

$$Y = \sum_{k=1}^{n} \delta_k[1-\cos\ (2k-1)\frac{\pi}{2}\ x\ ]$$

we could have obtained a value for the critical load very close to that given by (13) for sufficiently large value of n.

4.2 f(x) is a Polynomial in x.

Here the deflection curve is assumed as

$$Y = \delta_1 x + \delta_2 x^2 + \delta_3 x^3 \qquad \ldots (25)$$

So that the end conditions (9a) and (9b) are satisfied.

Using (25) and proceeding as in 4.1 we obtain equation (19), in which

$P \quad = \quad 288\ E_oI\ k^5/\ell^3,$

$k \quad = \quad (b/a)-1,$

$$I_{11} = 36h_1,$$

$$I_{22} = [16(k-z)^2 h_1 \pm 64(k-z)h_2 \pm 64h_3]/k^2,$$

$$I_{33} = [9(k^4-4k^3 + 10k^2 - 12k+9)h_1 + 36(k^3-5k^2+9k-9)h_2 + 18(5k^2-18k + 27)h_3 + 54(k-6)h_4 + 81h_5/k^4,$$

$$I_{12} = [24(k-2)h_1 + 48h_2]/k,$$

$$I_{23} = [12(k^3-4k^2 + 7k-6)h_1 + 24(2k^2-7k+9)h_2 + 12(7k-18)h_3 + 72h_4]/k^3,$$

$$I_{31} = 18[(k^2-2k+3)h_1 + (2k-6)h_2 + 3h_3]/k^2,$$

$$\beta_{11} = \frac{1}{4}, \quad \beta_{22} = \frac{1}{6}, \quad \beta_{33} = \frac{3}{20}, \quad \beta_{12} = \frac{1}{6},$$

$$\beta_{23} = \frac{3}{20}, \quad \beta_{13} = \frac{1}{8}, \quad \beta_{rs} = \beta_{sr} \text{ and } I_{rs} = I_{sr} = (r,s=1,2,3)$$

$$h_1 = [u^4(12 \log u + 23) - 48ku^3 - 36u^2 + 16u - 3)/12,$$

$$h_2 = [-4u^5 + 5ku^4 + 10u^3 - 10u^2 + 5u - 1]/5,$$

$$h_3 = [u^6 - 15u^4 + 40u^3 - 45u^3 + 24u - 5]/30,$$

$$h_4 = [u^7 - 35u^4 + 105u^3 - 126u^2 + 70u - 15]/105,$$

$$h_5 = [u^8 - 70u^4 + 224u^3 - 280u^2 + 160u - 35]/280.$$

Now, choosing papameters δ_r $(r=1,2,3)$ such that $(q)_{cr}$ is minimum and using similar arguments as in the previous case we obtain the value of a/b corresponding to the critical load as

$$\frac{a}{b} = 0.0210 \text{ approximately.}$$

Then by (7) the critical load is obtained as

$$(q\ell)_{cr} = 44.59 \frac{E_o I}{\ell^2} \qquad \text{... (26)}$$

Comparison of equation (26) with (24) and (13) suggests that equation (14) gives a closer approximation to the true deflection curve than that given by equation (25). Moreover, the approximate curve with two parameters ($\delta_3=0$) in (14) is even better than that with three parameters in (25).

5. CONCLUSION

The critical load for buckling in the associated homogeneous case is given by [Timoshenko and Gere, [2] p.103]

$$(q\ell)_{cr} = 7.837 \frac{E_o I}{\ell^2}$$

Comparison of this value with out present value given by (13) shows that the effect of non-homogeneity as given by (3) is to increase the critical load to about 5.08 times.

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GEOPHYSICAL PROSPECTING WITH LIGHTNING RADIATION

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ABSTRACT

A new method of geophysical prospecting using the lightning radiation is proposed. The detailed theoretical formulation shows that the ground conductivity profile can be plotted by recording the values of ratio of vertical to horizontal component of electrical field radiated from nearby lightning discharge (at a distance of 50 to 100 kms). This profile may be used for miniral exploration.

INTRODUCTION :

Use of natural fields for determination of ground conductivity dates back to 1953 when Cagniard [1] showed how the ratio of alternating magnetic field to alternating electric field plotted as a function of frequency can yield information about the variation of resistivity with depth within the earth. The method known as MT method measures simultaneously the magnetic and electrical oscillations caused by ionospheric currents and thunderstorms. These fields are recorded at different frequencies in the range from less than one hertz to 10 hertz. The same method using higher frequencies in audio range is known as AMT method.

Another closely related method was introduced in 1958 when Ward [2] measured the tilt of magnetic component of natural electromagnetic field of distant thunderstorms (at thousand of kms). At these large distances the field propagates in such a way that electric components are almost vertical and magnetic components almost hiorzontal. Presence of a conducting zone with

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a vertical boundary tilts the plane of polarisation and produces an appreciable vertical component of magnetic field. The tilt in (inclination of) the resultant magnetic field is measure of discountinuity in resistivity. This is known as AFMAG method as it uses the characteristics of audio fequency magnetic fields.

Another method on similar principle but using an artificially transmitted electromegnetic radiation is conductivity determination by wave-tilt measurements [3] where conductivity is determined from the ratio of horizontal component (directly measured) and vertical component of electrical field (determined by measuring horizontal magnetic field) emitted from VLF transmitter at a known frequency, power and distance from point of observation. Recently artificial source has been used for AMT method also [4].

These methods determine the ground conductivity and the data is interpreted according to required applications, e.g. search of metals and minirals or finding the depth of bedrock or in geothermal exploration. The first of these methods uses radiation from the source at unknown distance. The magnitude of the variation is indefinite. For long distance sources high sensitivity is needed. A near thunderstorm would disturb the conditions and even the instrument may be damaged. Moreover, the theory is not applicable to for the radiation from near thunderstorm or transmission line. This condition is also same for AFMAG method, though this method is more qualitative than quantitative. The wave-tilt measurements from artificial transmitter at known distance frequency and power are better but they too may be disturbed by thunderstorms and the theory uses many approximations [5].

In this paper a possibility of new method of geo-physical prospecting using natural source is analysed and its use for different purposes is proposed.

LIGHTNING AS NATURAL TRANSMITTER :

Lightning is a heavy current transient electric discharge between two opposite charge centres within a thunderstorm or between lower charge of a thunderstorm and the ground. It results in a transfer of charge from one place to another in form of heavy currents with peak value of order of few tens of killo amperes. This time varying current through air column flows for finite time ($\sim$ 200 μs) and produces an oscillating dipole consisting of the current channel of varying length and its image in finitely conducting ground. Experimentally observed radiation from lightning extends over whole range of electromagnetic radiation beginning from few hertz in ELF to UHF and microwave besides the visible and infrared radiation. The maximum power in radio wave range is in VLF region and peaks between 5 kHz and 10 kHz. Thus the lightning is a strong natural source of VLF radiation. This VIF radiation can be used for geophysical prospecting if the details of the influence of ground conuctivity on the radiated fields is worked out. This is discussed in next section.

GROUND CONDUCTIVITY DEPENDENCE OF LIGHTNING RADIATION :

Pathak et al. [6] showed that the VLF radiation from lightning may be accounted for by radiation emitted from an oscillating dipole consisting of lightning channel and its image in the ground. They calculated the electric and magnetic field components by formulating a vector potential associated with lightning channel and an image vector potential associated with the image of the channel. The image vector potential and therefore the resultant vertical fields are functions of ground conductivity. It can be shown [7] that the radiation from a return stroke (vertical channel from ground to cloud) is independent of ground conductivity while radiation from other strokes like K-changes (vertical or slightly inclined channel within the cloud) or horizontal discharges depends largely on ground conductivity. Physically these results are understood from the following well known facts: (i) A current channel (with any arbitrary orientation) over a perfectly conducting ground (hypothetical case) produces only vertical

electrical fields (ii) An imperfectly conducting ground develops a horizontal component of electric field in two ways : (a) Horizontal currents perpendicular to the plane of incidence (a vertical plane containing source and receiver) produce a horizontal E perpendicular to the plane of incidence. (b) Vertical currents at sufficiently great heights and horizontal currents in the plane of incidence produce a horizontal E parallel to the plane of incidence. Thus the return stroke does not produce horizontal electric component and therefore the vertical component from return stroke remains independent of ground conductivity. In other cases like K-change and horizontal channel, a horizontal component is developed because of penetration of a part of energy into the earth and making reflection coefficient less than unity. As the reflection coefficient, hence the penetration depends upon the ground conductivity, the horizontal and vertical E field also become the functions of conductivity.

To obtain the analytical expressions for vertical and horizontal electric fields a vector potential **A** is formulated from the channel current I as follows:

$$A(t)=\frac{\mu_o}{4\pi}\int\frac{J(t')}{r}dv' = \frac{\mu_o}{4\pi}\int\frac{I(t')}{r}\mathbf{dz}' = \frac{\mu_o}{4\pi}\frac{o}{r}F(t') \quad ...(1)$$

where t' ($= t - \frac{r}{c}$) is rerarded time, $\mathbf{dz}'$ is retarded elemental length along the channel and A_o a vector constant in time. The distance of observation point being larger than the channel dimensions is kept outside the integral. Assuming the origin of coordinate system at starting point of channel and z-axis vertically upward r, θ and ϕ- components of **A** can be written and electric and magnetic fields can be devaluated from the component equations of the relations

$$H = \frac{1}{\mu_o} \nabla \cdot X\, A \text{ and } E = \frac{1}{\varepsilon_o} \int \nabla X . H\, dt \qquad \ldots (2)$$

Similarly the field components due to image vector potential $A' = R_c A$, R_c being reflection coefficient, can be written and expressions for corresponding field components E_r'. E_θ'. E_ϕ' can be evaluated. But we need most commonly measured vertical and horizontal components of the fields. So the vertical electric field is obtained by summing all the vertical components of E_r, E_θ and E_r', E_θ', i.e.

$$E_v(t) = \frac{A_o}{4\pi\varepsilon_o} \left[\frac{f_1}{r^3} \int F(t')dt + \frac{f_1}{r^2 c} \frac{\partial F(t')}{\partial t} \right] \qquad \ldots (3)$$

For horizontal electric field, the horizontal components of E_r, E_θ, E_r, E_θ are added to obtain the field in the plane of incidence. The field perpendicular to the plane of incidence is the sum of E_ϕ and E_ϕ'. The resultant horizontal field is square root of the sum of the squares of fields in two perpendicular horizontal directions i.e.

$$E_h(t) = \frac{A_o}{4\pi\varepsilon_o} \Big[\left\{ \frac{f_3}{r^3} \int F(t')dt + \frac{f_3}{r^2 c} F(t') + \frac{f_4}{rc^2} \frac{\partial}{\partial t} F(t') \right\}^2 + \left\{ \frac{f_5}{r^3} \int F(t')dt + \frac{f_5}{r^2 c} F(t') + \frac{f_5}{rc^2} \frac{\partial}{\partial t} F(t') \right\}^2 \Big]^{\frac{1}{2}}$$

$$\ldots (4)$$

where,

$$f_1 = \cos\theta_1[2-\sin^2\theta+(3\sin^2\theta-2)R_v]-\sin\theta_1\sin\theta\cos\theta\cos(\phi_1-\phi)(1-3R_v) \quad \ldots (5)$$

$$f_2 = (1+R_v)[\cos\theta_1\sin^2\theta+\sin\theta_1\sin\theta\cos\theta\cos(\phi_1-\phi)] \quad \ldots (6)$$

$$f_3 = (1+3R_v)\cos\theta_1\sin\theta\cos\theta+\sin\theta_1[2-\cos^2\theta+(2-3\cos^2\theta)R_v]\cos(\phi_1-\phi) \quad \ldots (7)$$

$$f_4 = (1-R_v)[-\cos\theta_1\sin\theta\cos\theta+\sin\theta_1\cos^2\theta\cos(\phi_1-\phi)] \quad \ldots (8)$$

$$f_5 = -(1+R_h)\sin\theta_1\sin(\phi_1-\phi) \quad \ldots (9)$$

$$R_v = \frac{S-T}{S+T}\,;\quad R_h = \frac{\cos\psi-T}{\cos\psi+T} \quad \ldots (10)$$

with

$$S = [(\varepsilon+\frac{\sigma}{iw})/\varepsilon_0]\cos\psi;\quad T = [(\varepsilon+\frac{\sigma}{iw})/\varepsilon_0-\sin^2\psi]^{\frac{1}{2}} \quad \ldots (11)$$

and $\psi = \pi - \theta$. The reflection coefficients R_v and R_h are the functions of permittivity and conductivity of the ground. Therefore the electric field components are also the functions of these parameters.

THE PARAMETER TO BE MEASURED

Equations (3) and (4) give time-dependent values of vertical and horizontal electric field. These can be subjected to Fourier transformations to get the frequency dependent values as

$$E_{v,h}(w) = \int_0^{\infty} E_{v,h}(t)e^{-iwt}dt \qquad ...(12)$$

However, the field-components thus obtained are not the functions of w and σ only. They also contain lightning parameters through A_o. Eq(1) shows that A_o would contain lightning constants like peak value of current and decay constants. These vary from one discharge to another and not easy to determine. Simple way to get rid of these parameters is to consider ratio $R = E_v/E_h$ which would depend only on orientation of lightning discharge besides w and σ. Influence of orientation of the channel is very easy to visualise through eqs. (5)-(9), as follows:

For vertical discharge $\theta_1 = 0$ then

$$f_1^v = 2 - \sin^2\theta + R_v(3\sin^2\theta - 2)$$

$$f_2^v = (1+R_v)\, \text{Sin}^2\theta$$

$$f_3^v = (1+3R_v)\, \text{Sin}\theta \cos\theta \qquad ... (12)$$

$$f_4^v = -(1-R_v)\sin\theta\cos\theta$$

$$f_5^v = 0$$

For a vertical discharge starting from ground like return stroke; $\theta = \pi/2$, then $f_1 = f_2 = (1+R_v)$, $f_3 = f_4 = f_5 = 0$. This gives $E_h^v = 0$ and $R = \infty$ in accordance with the concepts that a vertical current near ground does not produce any horizontal field. Also for this case $|R_v| = 1$, so E_v becomes independent of ground conductivity which is obvious when the horizontal component is not developed i.e. energy ponetration into the ground is zero. Same result is obtained by Pathak and Hazarika [7].

For vertical discharge at sufficient height h above the ground $\theta = \pi/2 + \sin^{-1}(\frac{h}{r})$. The computed values of R at a fixed frequency of 5 kHz are plotted with ground conductivity for different values of r in Fig.1a.

For horizontal discharge; $\theta_1 = \pi/2$. But the channel may have any arbitrary inclination from the plane of incidence. If the channel is perpendicular to the plane of incidence i.e. $_1 - \quad = \pi/2$ so that $f_1 = f_2 = f_3 = f_4 = 0$ and $f_5^h = (1 + R_v)$, we have $E^h = 0$ and therefore $R = 0$. The field is confined to a plane perpendicular to the plane of incidence. For the case when channel is in the plane of incidence. $\phi_1 - \phi = 0$, so that

$$
\begin{aligned}
f_1^h &= (1 - 3R_v) \sin\theta \cos\theta \\
f_2^h &= (1 + R_v) \sin\theta \cos\theta \\
f_3^h &= 2 - \cos^2\theta + (2 - 3\cos^2\theta) R_v \qquad \ldots (13) \\
f_4^h &= (1 - R_v) \cos^2\theta \\
f_5^h &= 0
\end{aligned}
$$

This gives E_h^h confined to the plane of incidence as expected. Again $\theta = \pi/2 + \sin^{-1}(\frac{h}{r})$. The variation in the values of R is too small to be plotted or to be measured over the experimental error (see Table 1). The overall variation of R for arbitrary orientation of horizontal channel is between zero and 0.41. For requirement of high sensitivity and of determination of orientation $(\varphi_1 - \varphi)$, the horizontal channel is not recomended for use in determination of ground conductivity.

FIELD PROCEDURE :

As shown above the parameter depending upon minimum number of lightning parameter is ratio R of vertical and horizontal electric field. A non-zero, finite value of R with sufficient variation is obtained only for the case of intracloud vertical lightning. Field procedure would be to measure E_v and E_h at a fixed frequency (say 5 kHz). These are measured with the help of vertical and horizontal whip antinnas suplimented by tuned amplifiers. Amplification is to be sufficient to amplify a signal of a fraction of mV to the magnitude required by recorder. For measuring the resultant horizontal field, signals from two crossed whips are fed into a system of squaring, adding and square-rooting IC's connected as shown in Fig.1b. After

this stage E_v and E_h may be recorded separately and ratio may be obtained manually or a dividing IC may be used to obtain R directly. The distance of discharge is recorded from the time log Δt between electrical signal and thunder heard. Then r = Δt X speed of sound. If $E_v > E_h$ i.e. $R > 1$, the value of σ may be determined from Fig. 1a. By using many amplifieres tuned to different frequencies, average σ for different depths of penetration for these frequencies is plotted. This gives the conductivity profle with depth. This may be used for exploring the presence of minirals within the earth, and the distribution of ground water.

CONCLUSION :

From the fargoing discussion it is clear that by the observation of ratio of vertical and horizontal electic field radiated from nearly lightning (in range of distance of 50 to 100 km) the ground conductivity profile may be determined for the use in miniral exploration. The expressions are not valid for very near large distances (more than 100 km).

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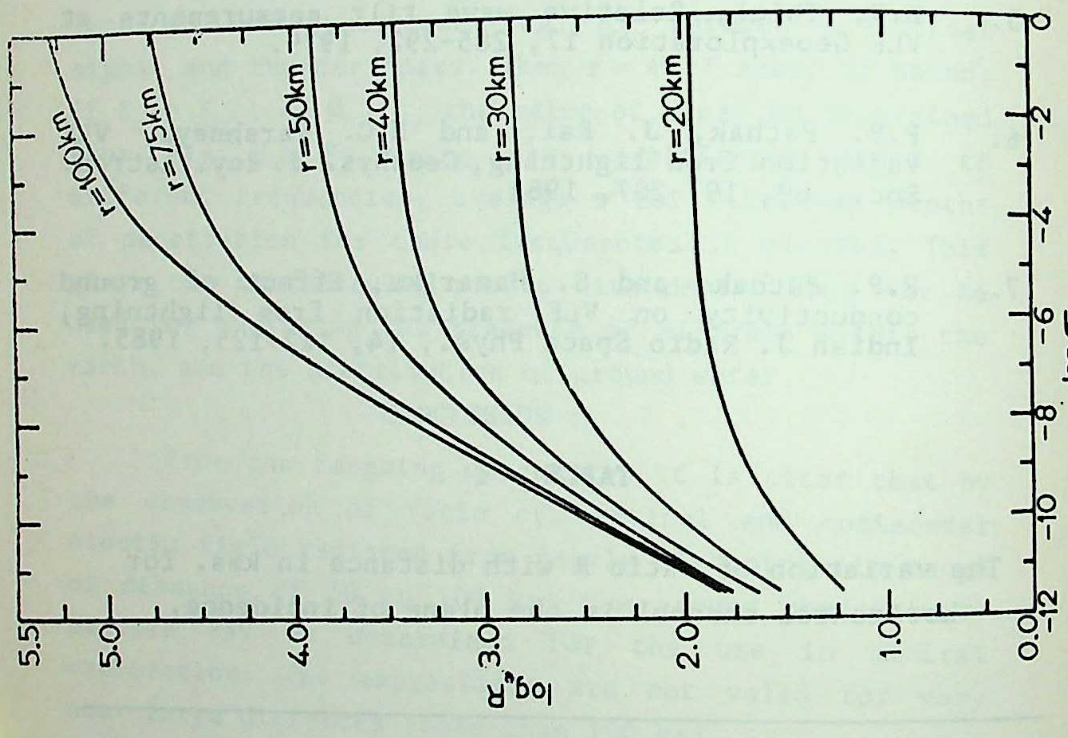
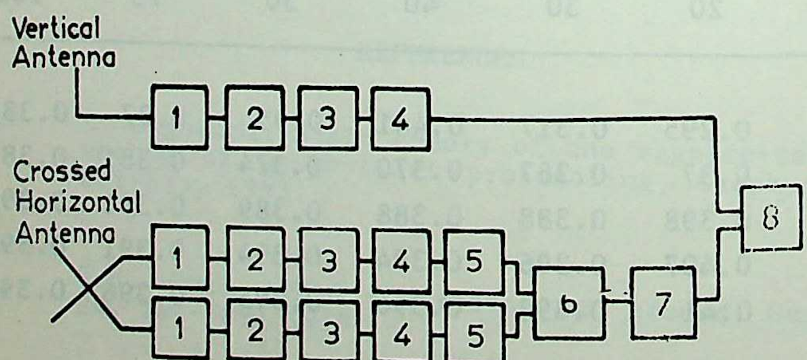
TABLE - 1

The variation of ratio R with distance in kms. for horizontal channel in the plane of incidence.

$\sigma(\Omega\,m)^{-1}$	20	30	40	50	75	100
10^{-5}	0.295	0.317	0.441	0.35	0.37	0.38
10^{-4}	0.37	0.367	0.370	0.374	0.381	0.385
10^{-3}	0.398	0.388	0.388	0.389	0.391	0.393
10^{-2}	0.407	0.396	0.394	0.394	0.391	0.396
10^{-1}	0.41	0.398	0.396	0.395	0.396	0.397





1. Matching Amplifier (Unity gain), 2. Tuner , 3. Voltage Amplifier,
4. Power Amplifier, 5. Squaring V.A., 6. Adding Circuit,
7. Square Rooting V.A., 8. Dividing Circuit (Digital Recorder)

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NOTE ON THE DEFLECTION IN A SIMPLY SUPPORTED TIMOSHENKO BEAM OF NONHOMOGENEOUS MATERIAL

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ABSTRACT

In expressions for the deflections in a simply supported beam of nonhomogeneous material have been obtained by using both technical and Timoshenko beam theories. It has been assess that the Young's modulus of the beam is a function of position and the beam is under the action of variable load. A comparative study has been made, to assess the effect of nonhomogeneity as has been considered in our note, in the tables for different values of the nonhomogeneity parameter.

Keywords : Timoshenko beam, Deflection, Nonhomogeneous material, Young's modulus.

INTRODUCTION :

The one dimensional theory of beams may have wider range of applicability provided the effects of transverse shear deformation, and the effects of rotatory inertia in the case of vibrating beams, are considered. When these effects are taken into account in the technical theory of beams, the new theory is known as Timoshenko beam theory [17, 18]. Dengler and Goland [6], Goland et

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al.[8], Miklowitz [11,12] studied the problems of dynamic response of Timoshenko beams. Achenback and Sun [1] obtained the steady state solutions of the response of the Timoshenko beam to a force moving with constant velocity over the beam which was supported by a deformable foundation. Ahmadi and Hashemi [2] studied the problem of vibration of an infinitely long Timoshenko beam under two types of random loadings. Chang and Juan [3] derived a set of equations for the free vibration of an inclined bar with an end constraint including the effect of rotatory inertia and shear deformation. Chen [4] presented the general dynamic stiffness matrix of a Timoshenko beam for transverse vibration. Among the many other problems of the dynamic response of Timoshenko beams mention may be made of the interesting works of Howson and Williams [9], Mead [10], Nayfeh [13], Park [14], Rao [15], Rossettos and Squires [16]. The statical Timoshenko beam problems have also received considerable attention in the literature. A comparative study, for the deflections in a uniformily loaded simply supported elastically homogeneous beam following the above theories, has been given in Dym and Shames [7].

As we know that there are plenty of materials in nature which are not elastically homogeneous and as it has been experimentally verified that in most of the cases the elastic co-efficients should not be taken as uniform throughout the beam, the purpose of the present investigation is to extend the foregoing studies given in Dym and Shames [7] to a class of nonhomogeneous beams subjected to nonuniform loading. The nonhomogeneity has been supposed to be arising from the variation of

Young's modulus with position. The general expressions for the deflections have been given. Numerical evaluations of deflections at different positions of the beam have been carried out to study the effects of shear deformation at a glance. Results of the corresponding homogeneous cases indicate that nonhomogeneity has significant effect on deflection.

FORMULATION OF THE PROBLEM AND ITS SOLUTION

(a) Solution based on the technical theory of beams

Choosing x-axis along the axis of the beam, let the position of the beam be given by $a \leqslant x \leqslant 2a$ ($a > 0$). The equation of equilibrium for the bending of the beam is given by

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = qx^2 \quad \ldots (1)$$

where w represents the deflection of the centreline of the beam under the action of variable load q(x) on the centroidal axis of the beam. E is the variable Young's modulus and I is the moment of inertia of the cross section of the beam about an axis lying in its plane.

We assume that the Young's modulus of the beam is a function of position according to the law,

$$E = E_o e^{-m(\frac{x}{a})} \quad \ldots (2)$$

and loading depends upon the position following the relation,

$$q = q_o \left(\frac{x}{a}\right)^n \qquad \ldots (3)$$

where E_o, q_o, m and n are real constants.

As regards the boundary condition of the problem we assume that both ends of the beam are simply supported. Mathematically speaking,

$$w = 0$$

$$\text{and} \quad \frac{d^2 w}{dx^2} = 0 \text{ at } x = a \text{ and } x = 2a. \qquad \ldots (4)$$

Introducing the nondimensional variable $X = \frac{x}{a}$ and taking into account of (2) and (3), field equation (1) becomes,

$$\frac{E_o I}{a^4} \frac{d^2}{dX^2}\left[e^{-mX} \frac{d^2 w}{dX^2}\right] = q_o X^n \qquad \ldots (5)$$

and the boundary condition (4) takes the form,

$$w = 0$$

$$\text{and} \quad \frac{d^2 w}{dX^2} = 0 \text{ at } X = 1 \text{ and } X = 2. \qquad \ldots (6)$$

Thus our problem is to solve (5) subject to (6).

Integrating equation (5) twice with respect to X and making use of the boundary condition $\frac{d^2 w}{dX^2} = 0$ at $X = 1$ and $X = 2$ we obtain,

$$\frac{d^2 w}{dx^2} = A_1 X^{n+2} e^{mX} + B_1 X e^{mX} + C_1 e^{mX} \qquad \ldots (5a)$$

finally integrating (5a) twice with respect to X and making use of the boundary condition w = 0 at X = 1 and X = 2 we obtain the expression for w as

$$w = w_o(X-1) + A_1 f(X) + B_1 g(X) + C_1 \theta(X) \qquad \dots (7)$$

where

$$f(X) = \int_1^X \left(\int_1^u t^{n+2} e^{mt} dt \right) du$$

$$g(X) = \int_1^X \left(\int_1^u t\, e^{mt}\, dt \right) du$$

$$\theta(X) = \int_1^X \left(\int_1^u e^{mt} dt \right) du$$

$$A_1 = \frac{a^4}{E_o I} \frac{q_o}{(n+1)\ (n+2)}$$

$$B_1 = A_1 (1 - 2^{n+2})$$

$$C_1 = A_1 (2^{n+2} - 2)$$

and w_o is the value of $\frac{dw}{dX}$ at X = 1 and it satisfies the equation

$$w_o + A_1 f(2) + B_1 g(2) + C_1 \theta(2) = 0. \qquad \dots (8)$$

(b) Solution based on the Timoshenko beam theory

The method discussed in the previous section does not indclude the effects of shear deformation. For short stubby beams, however, it is natural that this contribution cannot be neglected. Hence arises, the question of considering Timoshenko theory of beams. In this theory it is assumed that line elements normal to the centreline of the beam in the undeformed state move only in the vertical direction and also remain vertical during deformation. Line elements tangent to centreline undergo a rotation β (x) and satisfy the relation,

$$\frac{dw}{dx} = \Psi(x) + \beta(x), \qquad \text{... (9)}$$

$\frac{dw}{dx}$ being the total slope of the centreline and $\Psi(x)$ is the rotation of the line elements due to bending only.

In this case we get the two field equations as,

$$\frac{d}{dx}\left[EI\frac{d\Psi}{dx}\right] + KGA\left(\frac{dw}{dx} - \Psi\right) = 0 \qquad \text{... (10)}$$

and $$\frac{d}{dx}\left[KGA\left(\frac{dw}{dx} - \Psi\right)\right] + q = 0 \qquad \text{... (11)}$$

where G is the shear modulus, K is the shear coefficient and A is the area of cross section.

Differentiating (10) with respect to x and using (11) we get the single field equation as

$$\frac{d^2}{dx^2}\left[EI\,\frac{d\Psi}{dx}\right] = q. \qquad \text{... (12)}$$

In terms of dimensionless variable $X = \frac{x}{a}$ the field equation (12) when (2) and (3) are taken into account, becomes ,

$$\frac{E_o I}{a^3} \frac{d^2}{dX^2} \left[e^{-mX} \frac{d\Psi}{dx} \right] = q_o X^n \qquad \ldots (13)$$

and the boundary conditions become

$$w = 0$$

and $\frac{d\Psi}{dX} = 0$ at $X = 1$ and $X = 2$. ... (14)

Following the same method as discussed in the previous section the solution of (13) satisfying (14) is obtained as,

$$w = a\,\Psi_o (X-1) + A_1 f(X) + B_1 g(X) + C_1 \theta(X) - D_1\, \Phi(X) \quad \ldots (15)$$

where $\phi(X) = (n+2) \int_1^X t^{n+1}\, e^{mt}\, dt + (1-2^{n+2}) \int_1^X e^{mt}\, dt$

and $D_1 \qquad A_1 \frac{12+11\nu}{60} (\frac{h}{a})^2$.

In deriving (15) we have replaced G by $E/2(1+\nu)$ and A by bh. We have also introduced $I = \frac{1}{12} bh^3$, where b and h are the dimensions of the rectangular cross section of the beam and we have chosen the value of shear co-efficient K for a rectangular cross section from a paper by Cowper [5] as,

$$K = \frac{10(1+\nu)}{12+11\nu}$$

ν being the constant Poisson's ratio.

The quantity Ψ_o appearing in equation (15) is the value of Ψ at $X = 1$ and it satisfies the equation,

$$a\Psi_o + A_1 f(2) + B_1 g(2) + C_1 \theta(2) - D_1 \phi(2) = 0. \qquad \ldots (16)$$

Comparing equation (15) with equation (7) we see that the effect of shear deformation on the deflection in the nonhomogeneous beam lies in the last term in equation (15).

NUMERICAL RESULTS

To study the effect of nonhomogeneity as stipulated in (2) and the effect of nonuniform loading of the type (3) we have computed deflections for different values of m and n on taking Poisson's ratio $\nu = 1/3$. Deflection valuee have been computed at $X = 1.1\ (.1)2$ and the values of $R = \frac{E_0 I}{q_0 a} 4w$ for technical beam theory as well as for Timoshenko beam theory have been displayed in Tables 1 to 4 to consider different cases of interest. Moreover, to study the effects of shear deformation for increasing values of h/a the deflections for technical as well as shear theory have been shown for a particular position of the beam in Table 5.

Table-1

Values of R for m = 0,n = 0 and h/a = .1

X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.004087	.007733	.010587	.012400	.013020	.012400	.010587	.007733	.004087	0
R_{Timo}	.004205	.007942	.010862	.012713	.013347	.012713	.010862	.007942	.004205	0

Table - 2

Values of R for m = 1, h/a = .1 and different n

n = 0	X = 1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.017017	.032687	.045709	.054894	.059236	.058008	.050881	.038066	.020486	0
R_{Timo}	.017453	.033510	.046860	.056295	.060791	.059601	.052369	.039277	.021216	0
n = .5	X = 1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.020659	.039745	.055704	.067078	.072600	.071316	.062743	.047069	.025387	0
R_{Timo}	.021173	.040726	.057090	.068784	.074313	.073295	.064611	.048604	.026319	0
n = 1	X = 1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.025221	.048592	.068250	.082402	.089445	.088131	.077772	.058503	.031622	0
R_{Timo}	.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0
n = 1.5	X = 1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.030958	.059729	.084063	.101748	.110758	.109459	.096883	.073081	.039587	0
R_{Timo}	.031688	.061149	.086108	.104314	.113694	.112559	.099867	.075586	.041140	0
n = 2	X = 1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.038204	.073803	.104071	.126267	.137825	.136609	.121273	.091732	.049799	0
R_{Timo}	.039083	.075528	.106574	.129436	.141486	.140512	.125069	.094950	.051816	0

Table - 3

Valuesof R for $\bar{n}$ = 1, h/a = .1 and different m

	X =									
m = 0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.006004	.011403	.015691	.018485	.019531	.018714	.016071	.011797	.006258	0
R_{Timo}	.006165	.011695	.016084	.018945	.020021	.019195	.016501	.012131	.006450	0
m = .5	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.012245	.023431	.032588	.038877	.041641	.040461	.035216	.026161	.014004	0
R_{Timo}	.012551	.024006	.033386	.03984	.042701	.041535	.036209	.026960	.014478	0
m = 1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.025221	.048592	.068250	.082402	.089445	.088131	.077772	.058503	.031622	0
R_{Timo}	.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0
m = 1.5	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.052475	.101717	.144163	.176031	.193558	.193371	.173043	.131880	.072029	0
R_{Timo}	.053749	.104229	.147828	.180697	.198983	.199197	.178756	.136767	.075121	0
m = 2	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
R_{Tech}	.110298	.214942	.307145	.379021	.421948	.427315	.387815	.299568	.165430	0
R_{Timo}	.113087	.220521	.315423	.389760	.434701	.441328	.401904	.311949	.173491	0

Table - 4

Value of R for m = 1, n = 1 and different h/a

X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
h/a=.1										
R_{Timo}	.025831	.049769	.069929	.084488	.091809	.090603	.080128	.060460	.032823	0
X =	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
h/a=.01										
R_{Timo}	.025227	.048604	.068267	.082423	.089486	.088156	.077795	.058523	.031634	0

Table - 5

Values of R for m = 1, n = 0, X = 1.5 and different h/a

$R_{Tech} = 0.059236$

h/a	R_{Timo}
0.1	0.060791
0.2	0.065458
0.3	0.073236
0.4	0.084124
0.5	0.098123

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COMMON FIXED POINT THEOREMS IN METRIC LINEAR AND UNIFORM SPACES

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In this paper we generalize and improve a fixed point theorem in normed linear spaces due to Khan, Imdad and Sessa [3] to matric linear spaces and also prove a common fixed point theorem in uniform spaces.

Let (X,d) be a metric linear space. It has ball conveity if $r \in R^+$, x, $y \in X$ with $d(x,0) = d(y,0) = r \Longrightarrow d(\frac{x+y}{2}, 0) \le r$.

In was observed by Sastry and Naidu [6] that in a metric linear space (X,d) with ball convexity (B.C.), the function d(tx, 0) is a increasing function to t on R^+ for any x in X. We first observe the following:

LEMMA 1. Suppose (X, d) is a metric linear space with ball convesity, $\{a_n\}$ is a sequence in X and $\{\alpha_n\} \subseteq [0, 1]$ such that $\{\alpha_n\}$ is bounded away from zero. If $\{\alpha_n a_n\}$ converges to zero then $\{a_n\}$ also converges to zero.

Proof. Since $\{\alpha_n\}$ is bounded away from zero, there exists a positive integer N such that $\alpha_n > \frac{1}{N}$ for sufficiently large n. Since d(tx, 0) is an increasing function of t or R^+ for any x in X, we have $d(a_n,0) = d(\frac{1}{\alpha_n} \alpha_n a_n, 0) \le d(N\alpha_n a_n,0)$, for sufficiently large n,

$\le N d(\alpha_n a_n, 0)$ since d is translation invariant.

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Since $\{\alpha_n a_n\}$ converges to zero it follows that $\{a_n\}$ also converges to zero.

Definition (Jungck [2]). Two self-maps f and g on a metric space (X,d) are said to be compatible on X iff $\underset{n\to}{\mathrm{Lt}}\, d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\underset{n\to}{\mathrm{Lt}}\, fx_n = u = \underset{n\to}{\mathrm{Lt}}\, gx_n$ for some u in X.

THEOREM 2. Let (X,d) be a metric linear space with ball convexity and K be a nonempty convex subset of X; f, g, h be self-mappings of K satisfying

(2.1) $d(fx,gy) \le \emptyset(d(hx,hy), d(hx,fx)+d(hy,gy), d(hx,gy) + d(hy,fx), d(hx,fx)+d(hx,gy), d(hy,gy)+d(hy,fx))$

for all x, y in K where $\emptyset : R^{+5} \to R^{+}$ is an upper semi continuous function such that (i) $\phi(0, t, t, t, t) < t$ for $t > 0$, (ii) $\emptyset(t, 0, 2t, t, t) < t$ for $t > 0$.

Suppose there exists a sequence $\{C_n\}$ [0,1] which is bounded away from zero.

Suppose for $x_o \in K$, there exists a sequence $\{x_n\}$ such that $hx_{2n+1} = (1-C_n)\, hx_{2n} + C_n\, fx_{2n}$, $hx_{2n+2} = (1 - C_n)\, hx_{2n+1} + C_n\, gx_{2n+1}$ for $n \geq 0$. Suppose either f and h or g and h are compatible on K. If $\{hx_n\}$ converges to z in K and h is continuous at z then z is the unique fixed point of f, g and h.

Proof:- Clearly $hx_{2n+1} - hx_{2n} = C_n\,(fx_{2n} - hx_{2n})$.

By Lemma 1, $fx_{2n} - hx_{2n} \to 0$ as $n\to\infty$ and hence $fx_{2n} \to z$ as $n\to\infty$. Similarly $gx_{2n+1} \to z$ as $n\to\infty$.

Suppose f and h are compatible on K. then

$$d(fhx_{2n}, hfx_{2n}) \to 0 \text{ as } n\to\infty .$$

Since h is continuous at z we have $hfx_{2n} \to hz$, $h^2x_{2n} \to hz$ as $n\to\infty$ and hence $fhx_{2n} \to hz$ as $n\to\infty$. Now,

$$d(fhx_{2n}, gx_{2n+1}) \leq \phi\, (d(h^2x_{2n}, hx_{2n+1}), d(h^2x_{2n}, fhx_{2n}) + d(hx_{2n+1}, gx_{2n+1}), d(h^2x_{2n}, gx_{2n+1}) + d(hx_{2n+1}, fhx_{2n}), d(h^2x_{2n}, fhx_{2n}) + d(h^2x_{2n}, gx_{2n+1}), d(hx_{2n+1}, gx_{2n+1}) + d(hx_{2n+1}, fhx_{2n})).$$

Letting $n \to \infty$, we get

$$d(hz, z) \leq \phi\, (d(hz, z), 0, 2\, d(hz, z), d(hz, z)).$$

By (ii), we get $hz = z$.

Suppose g and h are compatible on K.

Then putting $x = x_{2n}$, $y = hx_{2n+1}$ in (2.1) and letting $n \to \infty$ and using (ii) one can prove that $hz = z$.

Now,

$$(2,2)\quad d(fz, gx_{2n+1}) \leq \phi\, (d(hz, hz_{2n+1}), d(hz, fz) + d(hx_{2n+1}, gx_{2n+1}), d(hz, gx_{2n+1}) + d(hx_{2n+1}, fz), d(hz, fz) + d(hz, gx_{2n+1}), d(hx_{2n+1}, gx_{2n+1}) + d(hx_{2n+1}, fz)).$$

Letting $n \to \infty$, we get

$$d(fz, z) \leq \phi\, (0, d(z, fz), d(z, fz), d(z, fz)\; d(z, fz)).$$

By (i), we get $fz = z$.

Similarly putting $x = x_{2n}$, $y = z$ in (2.1) and letting $n \to \infty$ and using (i), one can prove that $gz = z$.

Thus z is a common fixed point of f, g and h. uniqueness of common fixed point follows easily from (2.1) and (ii).

COROLLARY 3. Theorem 2 holds if the inequality (2.1) is replaced by

(3.1) $d(fx, gy) \leq a\, d(hx, hy) + b\, [d(hx, fx) + d(hy, gy)]$

$+ c\, [d(hx, gy) + d(hy, fx)]$

for all $x, y \in K$ where $a + b + c \geq 0$, $b + c < 1$, $a + 2c < 1$.

Proof . Corollary 3 follows by selecting $\emptyset(t_1, t_2, t_3, t_4, t_5) = at_1 + bt_2 + ct_3$ in Theorem 2.

REMARK 4 . In Corollary 3, the assumption '$\{hx_n\}$ converges to z in K' cannot be dropped even when $g = f$, h = identify map, $C_n = 1$ for all n in view of the following example.

Example. Define $F ; R_+ \rightarrow R_+$ as $F(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ 1, & t \geq 1 \end{cases}$

Then (R, d), where $d(x, y) = F(|x-y|)$ for all $x, y \in R$, is a Complete matric linear space with ball convexity.

Let $K = [0, \infty)$. Define $f : K \rightarrow K$ as $fx = x+1$. Then $d(fx, fy) \leq \frac{1}{2} d(x,y) + \frac{1}{4} [d(x, fx) + d(y, fy)]$ for all $x, y \in K$.

Here $a = \frac{1}{2}$, $b = \frac{1}{4}$, $c = 0$.

For any $x_o \in K$, the sequency $\{x_n\}$ defined by $x_{n+1} = fx_n$, $n = 0, 1, 2, \ldots$ does not converge to any $z \in K$.

f has no fixed point.

REMARKS. The following example shows the situation where Corollary 3 is applicable and Theorem 7 is not applicable even when $g = f$, $C_n = 1$ for all n.

Example. Let $X = K = (0, \infty)$, $d(x, y) = |x - y|$ for all $x, y \in K$. Let $fx = x^2$, $hx = 2x^2$ for $x \in K$. Clearly $f(X) = h(X)$ and $d(fx, fy) = \frac{1}{2} d(dx, hy)$ for all $x, y \in K$.

Clearly f and h are not weakly commuting. But f and h are compati[illegible].

For $x_o = \frac{1}{2}$ there exist $x_1 = \frac{1}{2\sqrt{2}}$, $x_2 = \frac{1}{2^2}$, $x_3 = \frac{1}{2^2\sqrt{2}}$, ...

such that $hx_{n+1} = fx_n = \frac{1}{2^{n+2}} \to 0 \in K$.

Clearly h is not linear. Zero is the only common fixed point of f and h.

REMARKS 6. In Corollary 3, the assumption "h is continuous at $z \in K$" cannot be dropped even when $g = f$, $C_n = 1$ for all n in view of the following example.

Example . Let $X = [0,1] = K$, $d(x,y) = |x-y|$ for all $x,y \in K$. Define $fx = \frac{x}{2}$ for all $x \in K$;

$hx = x$ if $x \neq 0$, $h(0) = 1$. Here $f(X) \not\subset h(X)$, h is not linear.

Clearly $d(fx, fy) \leq \frac{1}{2} d(hx, hy) + \frac{1}{2} [d(hx,fx) + d(hy,fy)]$ for all $x, y \in K$. f and h are weakly commuting and hence compatible. For $x_o \neq 0$ there exists $\{x_n\}$ such that $hx_{n+1} = fx_n \to 0 \in K$. Clearly h is not continuous at 0. But 0 is not the common fixed point of f and h.

Corollary 3 is an improvement of the following theorem of [3].

THEOREM 7 ([3]). Let f and h be two self-mappings of a linear normed space X such that f, h is weakly commuting pair, h is continuous, linear and $f(X) \subseteq h(X)$. Further, we have

$$||fx - fy|| \leq a\, ||hx - hy|| + b[\,||hx - fx|| + ||hy - fy||\,]$$
$$+ c\,[\,||hx - fy|| + ||hy - fx||\,]$$

for all $x, y \in X$ where $a + b + c \geq 0$, $a + 2b + 2c \leq 1$. Let $x_o \in X$ and $\{C_n\}$ be a sequence satisfying $0 < C_n \leq 1$ for $n \geq 0$ and $\operatorname{Lt}_{n\to\infty} C_n > 0$. Suppose the sequence $\{hx_n\}$ defined by $hx_{n+1} = (1 - C_n)\, hx_n + C_n\, fx_n$ (for $n \geq 0$) converges to a point z in X. Then z is a coincidence point of f and h. If $b > 0$ then fz is the unique common fixed of f and h.

In the proof of Theorem 7, they [3] used $a \geq 0$, $b \geq 0$ without mentioning it in the hypothesis for the existence of coincidence point.

REMARK 8. Theorem 2 is true if either f and h or g and h are compatible and h is continuous at z are replaced by either $fh = hf$ or $gh = hg$ and h^k is continuous at z for some positive integer k.

One can prove the theorem in Remark 8 by the similar lines of cases (7) and (2) in Theorem (1.1 of Naidu and Prasad [5].

Now we prove a theorem in uniform spaces.

Let (X, U) be a Hausdorff sequentially complete uniform space and let P be a fixed family of pseudo-metrics on X which generates the uniformity U. For any pseudo-metric p on X and $r > 0$, let $V_{(p,r)} = \{(x,y) \mid x,y \in X,\ p(x,y) < r\}$.

Let $G = \{V_n \mid V_n = \bigcap_{i=1}^{n} V_{(pi,\ ri)},\ r_i > 0,\ p_i \in P,\ i = 1, 2, \ldots\ldots, n\}$.

Two self-maps f and h in (X,U) said to be compatible if $p(fhx_n, hfx_n) \to 0$ as $n \to \infty$ whenever $\{x_n\}$ is a sequence such that $\operatorname{Lt}_{n\to\infty} fx_n = u = \operatorname{Lt}_{n\to\infty} hx_n$ for some u in X where p is any pseudo-metric on X.

THEOREM 9. Let f and h be self-maps of X such that h is continuyous, $f(X) \subseteq h(X)$ and f and h are compatible on X. Also for any V_1, V_2, V_3, V_4, V_5 in G and x, y in X, $(hx, hy) \in V_1$, $(hx, fx) \in V_2$, $(hy, fy) \in V_3$, $(hx, fy) \in V_4$, $(hy, fx) \in V_5$ implies

(9.1) $(fx, fy) \in a_1(x, y)\, V_1 \circ a_2(x,y)\, V_2 \circ a_3(x,y)\, V_3 \circ a_4(x,y)\, V_4 \circ a_5(x, y)\, V_5$

where $a_i : X \times X \to R^+$ for $i = 1, 2, 3, 4, 5$ such that

$$\sup_{x,y \in X} \sum_{i=1}^{5} a_i (x,y)] = \lambda < 1.$$

Then f and g have a unique common fixed point z in X.

Proof. Let x, y be arbitrary in X and V be arbitrary in G. Let p be the Minkowski pseudo-metric of V and put $p(hx,hy) = r_1$, $p(hx,fx) = r_2$, $p(hy,fy) = r_3$, $p(hx,fy) = r_4$, $p(hy,fx)=r_5$. Then with arbitary > 0, we have

$(hx,hy) \in (r_1 + \varepsilon) V$, $(hx,fx) \in (r_2+\varepsilon) V$, $(hy,fy) \in (r_3+\varepsilon) V$,

$(hx,fy) \in (r_4+ \varepsilon) V$, $(hy,fx) \in (r_5+\varepsilon) V$.

Write $a_i(x, y) = a_i$ for the sake of convenience.

Therefore by (9.1), we have

$$(fx,fy) \in a_1(r_1+\varepsilon)V_o\, a_2(r_2+\varepsilon)V_o\, a_3(r_3+\varepsilon)V_o\, a_4(r_4+\varepsilon)V_o\, a_5(r_5+\varepsilon)V$$

Since ε is arbitrary we have

$p(fx, fy) \leq a_1\, p(hx,hy) + a_2\, p(hx,fx) + a_3\, p(hy,fy) + a_4 p(hx, fy) + a_5 p(hy,fx)$ and it follows that

(9.2) $p(fx,fy) \leq \lambda \max \{p(hx,hy), p(hx,fx), p(hy,fy),$

$p(hx,fy), p(hy,fx)\}$ for all x, y in X.

For x_o in X, define $\{x_n\}$ such that $fx_n = hx_{n+1}$ for $n=0,1,2,\ldots$ It can be possible since $f(X) \subseteq h(X)$.

One can prove, by following the procedure as in Das and Naik [1] or Naidu and Prasad [4], that $\{fx_n\}$ is Cauchy. Hence $\{fx_n\}$ converges to some $z \in X$. By continuity of h and compatibility of f and h we get $fhx_n \to fx$ as $n \to \infty$.

Using (9.2) to the terms $p(fhx_n, fz)$ and $p(fhx_n, fx_n)$ and letting $n \to \infty$ we get $p(hz,fz) = 0 = p(hz, z)$.

So (hz, fz) and (hz, z) are in V for all V in G. Hence $hz = fz = z$. Uniqueness of common fixed point follows easily by (9.2).

REMARK 10. Theorem 9 is true if f and h are compatible and h is continuous are replaced by $fh = hf$ and h^k is continuous for some positive integer k.

Remark 10 is an improvement of Theorems 2.1, 3.1, 3.2, 4.2 of [1].

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STUDIES ON SOIL CONSERVATION VALUE OF COMMON PLANTS OF SHIVALIKS

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ABSTRACT

In the present study soil conservation value of *Cynodon dactylon*, *Adhatoda vasica*, *Ipomea carnea* *Cymbopogon martimii* and *Sida sps* has been investigated and it has been found that *Cynodon dactylon*, *Adhatoda vasica* and *Cymbopogon martimii* are the suitable species for plantation on hilly slopes because thier soil conservation value is maximum, being 82.5%, 72.1% and 48.4%, respectively. The moisture content of soil which helps in formation of soil texture under these plant species was also higher being 18.5%, 20.5% and 15.2%, respectively, (Fig.1).

Key words : Soil conservation value, soil moisture, leaf area index.

INTRODUCTION

Danger is looming large in Shivalik, known as a delicate (Kuchcha) mountain range. According to the botanists the 1000 feet high mountain surrounding the famous Mansa Devi temple faces collapse in a few years as a result of induscriminate cutting of plants, pilgrims activties and uncontrolled grazing. People felled trees at Bilkeshwar too build a Bilkeshwar colony. Not only did they cut valuable tree but even diverted a stream into the Ganga making themselves susceptible to land slide and floods. These hills are surrounded by many villages. The grazing animals as well as wood and fodder collectors of tnese villages reduce the forest wealth. It leads to loss

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of organic layer and humus which is helpful in conserving soil moisture. It is estimated that approximately 40 crore animals grage in Indian forests. These result in soil erosion on hilly slopes in many parts of India converting them into barrenlands. Shivaliks and Aravali hills of India are facing severe ill effects of uncontrolled grazing by heeds of cattle, Maheshwari [6]. Soil erosion from Shivalik in Hardwar and river bank in downstream was found to the enormous during rains. Ambasht [1] has found the soil conservation values of *Euphorbia munja* (92-96%), *Cynodon dactylon* (90-97%) and Weed *Euphorbia hirta* (10-12%). Besides, several grasses have been found suitable for checking soil erosion in the Nilgiri hill e.g. *Eragrostis amabitis*, *E. cervula*, *Cynodon dactylon* and *Dactylis glomerata*,, Bhatia [3]. Hence, to check soil erosion, the most effective and natural process would be to plant suitable species at suitable sites, Shanker [7].

MATERIALS AND METHODS

The selection of plants has been done on the basis of their economic importance, root habit and ground cover. Plant were planted on the experimental sloping plots each of 1x3m in size (*In the G.Kangri botanical garden*) with an inclination angle of 13°. At the base of the slope cemented runoff collectors were constructed to receive the water runoff and eroded soil from the plots. Water was sprayed upon the plots from a height of 1mt for 30 min at a constant speed of 16l/min showering done at 15 days intervals, Ambasht [2].

(a) $$CV = 100 - \left(\frac{S_{wp}}{S_{wo}} \times 100\right)$$

where – CV = Conservation value

S_{wp} = Weight of soil washed from vegetated plots

S_{wo} = Weight of soil eroded from the bare plot

(b) LAI = TLA/TGA

where
LAI = Leaf area index
TLA = Total leaf area
TGA = Total ground area

RESULTS AND DISCUSSION

In the present study soil conservation value and soil moisture content of *Adhatoda vasica, Ipomea carnea, dactylon, Cymbopogon martimii & Sida sps* has been investigated. As evident from table (I) *Cynodon dact-ylon* and *Adhatoda vasica* showed higher conservation value i.e. 49.2% and 45.0% the moisture content of soil, which helps in formation of soil texture under these plants was also higher being 19.6% and 21.9%, respectively.

Ipomea carnea, Sida sps and *Cymbopogon martimii* showed lower conservation value standing at 25.8%, 37.8%, 37.6% and 39.3%, respectively. On day 15 the conservation value of *Cynodon dact-ylon* and *Adhatoda vasica* was found to be 68.9% and 52.6%, respectively and the moisture content of soil stood at 16.6% and 20.2%, respectively. *Ipomea carnea* , *Sida sps* and *Cymbopogon martimii* showed lower conservation value standing at 28.7%, 39.2% and 42.8%, while the moisture content of soil was also lower i.e. 15.4%, 15.8% and 13.7%, respectively. On day 30 also the maximum soil conservation value was recorded in the case of *Cynodon dactylon* and *Adhatoda vasica* standing at 75.5% and 68.0%, the moisture content of soil was also higher being 20.7% and 21.4%, respectively

The experiment was conducted for a period of 45 days. Best results in respect of soil conservation value and soil moisture content were obtained with *Cynodon dact-ylon* and *Adhatoda vasica* which on day 45 stood respectively at

82.5%, 18.5% and 72.1% and 20.2%. *Ipomea carnea*, *Sida sps* and *Cymbopogon martimii* showed soil conservation value and soil moisture content which on day 45 stood at 36.2%, 14.6%, 12.2% and 48.4%, 15.0% respectively. The result shows that relatively high soil conservation value and soil moisture constent as obtained under the effect of *Cynodon dact ylon* and *Adhatoda vasica* are attributable to better higher leaf area index of these plants i.e. 0.89 and 0.98 cm^2, respectively. Besides, thir root system also was better developed which could tightly bind the soil particles against erosion. The higher leaf area of these plants provided a cover to the soil against the impact of rain drops so that the soil particles were relatively mildly hit resulting in the checking of soil erosion. The present findings are supported by earlier work where it was found that the dense grass cover checks the velocity of the rain drops and result is more water entering the soil and no erosion because their canopy and profuse root system tightly bind the soil against erosion, Donahue [4]. Grasses can be grown on lands which are not suitable for cultivation of crops. They are also used in protecting bunds, water ways, badly eroded areas etc. Kanitkar et al.[5].

In the light of the present study *Cynodon dact-ylon* and *Adhatoda vasica* are recommended for growing in the Shivalik for soil and moisture conservation. *Adhatoda vasica* is an important medicinal plant and hence its plantation in the Shivalik shall add to the medicinal plant wealth of the region that has been dwindling over the years as a consequence of over exploitation and ecodegradation in the region.

TABLE-1

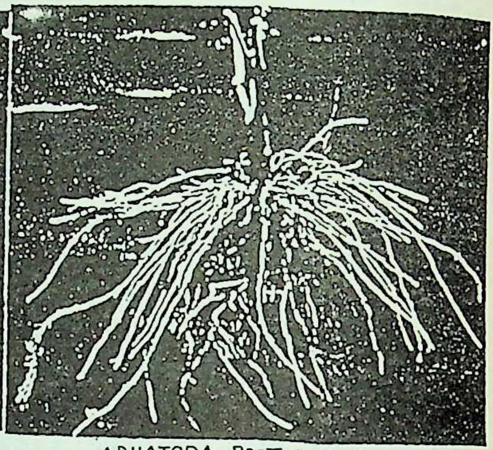
MOISTURE CONTENT AND SOIL CONSERVATION VALUE OF DIFFERENT PLANTS

INTERVAL DAY	NAME OF PLANT SPS.	AIR T °C	SOIL T °C	HEIGHT OF PLANT (Cm)	LAI (Cm^2) (mean)	SOIL MOISTURE %	SCV *%	% OF NITROGEN
0	A	24.0	22	-	-	11.1	NIL	0.010
	B	24.0	21	12.6	0.06	21.9	45.0	0.012
	C	24.0	22	14.3	0.031	18.3	25.8	0.012
	D	24.0	21.5	13.2	0.52	19.6	49.2	0.013
	E	24.0	18	14.0	0.026	14.9	39.3	0.013
	F	24.0	16.5	10.8	0.05	16.2	37.6	0.012
15	A	24.8	23.6	-	-	10.8	NIL	0.010
	B	24.8	21.8	18.7	0.49	20.2	52.6	0.012
	C	24.8	23.0	16.8	0.08	15.4	28.7	0.011
	D	24.8	21.6	16.3	0.63	16.6	68.9	0.013
	E	24.8	19.6	18.9	0.032	13.7	42.8	0.013
	F	24.8	16.8	12.4	0.09	15.8	39.2	0.011
30	A	28.6	26.0	-	-	8.7	NIL	0.008
	B	28.6	24.2	29.8	0.86	21.4	68.0	0.012
	C	28.6	27.8	38.2	0.38	16.8	32.7	0.011
	D	28.6	24.0	27.5	0.72	20.7	76.5	0.013
	E	28.6	26.7	29.2	0.036	15.2	46.6	0.014
	F	28.6	25.4	26.8	0.42	14.4	41.2	0.012
45	A	29.7	26.9	-	-	6.0	NIL	0.006
	B	29.7	24.0	34.8	0.98	20.2	72.1	0.013
	C	29.7	29.8	39.8	0.66	14.6	36.2	0.012
	D	29.7	24.5	28.4	0.89	18.5	82.5	0.013
	E	29.7	26.4	30.8	0.47	15.0	48.4	0.014
	F	29.7	26.0	28.8	0.56	12.2	45.6	0.013

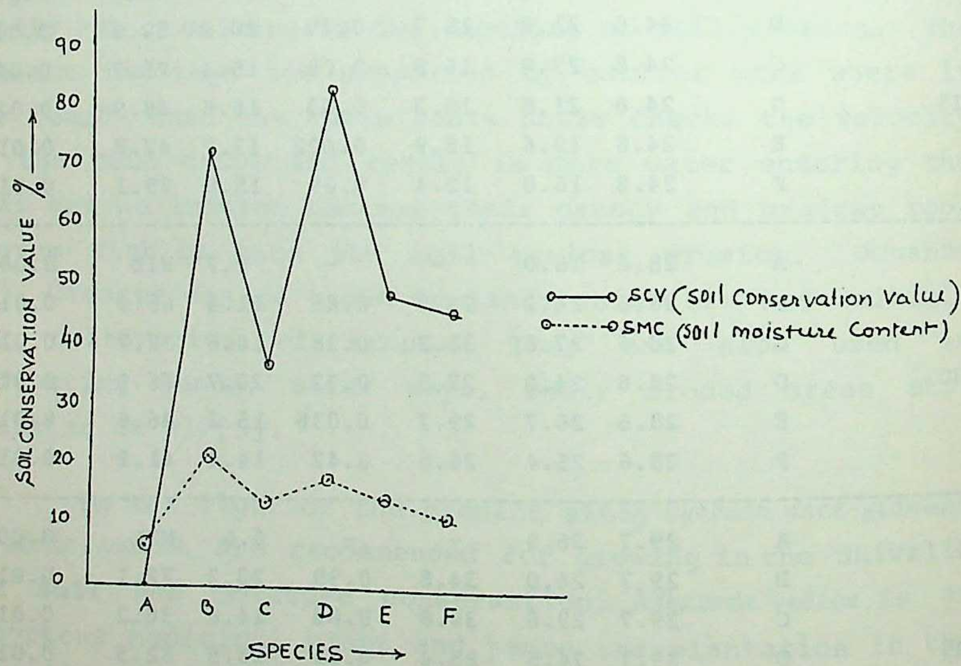
A = Control. B = *Adhatoda vasica* C = *Ipomea carnea* D = *Cynodon dactylon*.
E = *Cymobopogon martimii* F = *Sida sps*.

* Soil Conservation Value.

CYNODON ROOT HABIT



ADHATODA ROOT HABIT



(Fig 1)

A= control B= *Adhatoda vasica*, C = *Ipomea carnea*, D= *Cynodon dactylon*
E = *Cymbopogon martinii* F= *Sida sps.*

Fig 1: Graphical representation of SCV and SMC by different plant species

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SIGNIFICANCE OF THE CONSTANT TERM OF A HERMITE POLYNOMIAL OF EVEN DEGREE

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ABSTRACT

We present new proofs of the definite parity of the Hermite polynomials and the recurrence relations satisfied by them. We also show that *all* the Hermite polynomials will be simple polynomials if the constant term of *any* Hermite polynomial of even degree is non-zero.

Keywords and Phrases : Hermite polymonials, Wronskian, Parity, Recurrence relations, constant term of a Hermite polynomial of even degree.

Mathematics Subject Classification Number:33A65

INTRODUCTION

The purpose of this paper is to present new proofs of the definite parity of the Hermite polynomials [1,2,4] and their recurrence relations and also to look at the significance of the constant term of *any* Hermite polynomial of even degree. If $H_{2n}(0)$ is non-zero for a *single* positive integer $n \geq 0$, the entire set of the Hermite polynomials, $[H_m(x)]$, will be a simple set of polynomials, each polynomial in the set having a non-zero leading term and hence a precise

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degree [4]. The differential equation, recurrence relations, and zeros of the Hermite polynomials are the windows through which we look at the beauty of $H_{2n}(0)$, the constant term of $H_{2n}(x)$.

WRONSKIAN, PARITY AND RECURRENCE RELATIONS

The basis for our analysis is the differential equation satisfied by the Hermite polynomials [1,2,4]:

$$(D^2 - 2\,x\,D + 2p)\,H_p(x) = 0,\ p = 0,1,2,\ldots, \quad (1)$$

where $$D \equiv d/dx. \quad (2)$$

The Wronskian of the two solutions Y_1 and Y_2 of the Hermite differential equation (HDE), $W_{HDE}(x)$, obtained using Abel's formula [3,5,6], is given by

$$W_{HDE}(x) = Y_1Y_2' - Y_2Y_1' = W_{HDE}(0)\,\exp(x^2). \quad (3)$$

Since $W_{HDE}(x)$ is an *even* function of x and also an *infinite* series, the two linearly independent solutions of the HDE cannot be polynomials simultaneously. (The Wronskian of two polynomial solutions will be another polynomial, leading to $W_{HDE}(0) = 0$.) Moreover, the two linearly independent solutions of the HDE cannot have the same parity. (The Wronskian will be an odd function in this case, yielding $W_{HDE}(0) = 0$.) As (1) is a homogeneous second order linear differential equation, it has $H_p(x) \equiv 0$ as a trivial solution. But we are interested in finding a non-trivial polynomial solution satisfying (1). When $x \to -x$, $D \to -D$, $D^2 \to D^2$, and $-2xD \to -2xD$. Hence if $H_p(x)$ is a non-trivial polynomial solution of the HDE, $H_p(-x)$ is a also

solution of the same HDE. Since both $H_p(x)$ and $H_p(-x)$ are non-trivial polynomials, they are simply proportional to each other and hence $H_p(x)$ has a definite parity $(-1)^p$:

$$H_p(-x) = (-1)^p H_p(x). \tag{4}$$

We next obtain, from the HDE, the differential recurrence relation satisfied by the Hermite polynomials. From (1), it is clear that $D(D^2-2xD+2p)\ H_p(x) = D\ 0 = 0$. Hence we have.

$$[D^2-2xD+2(p-1)]\ [DH_p(x)=0, p\geq 1. \tag{5}$$

Thus both $H_{p-1}(x)$ and $DH_p(x)$, $p \geq 1$, are solutions of the same HDE. Since both are non-trivial polynomials (they have the same parity too), they cannot be the linearly independent solutions of the HDE. Hence $DH_p(x) = C_p\ H_{p-1}(x)$, $p \geq 1$, C_p being the constant of proportionality. The familiar choice is $C_p = 2p$, for all $p \geq 1$. (The leading coefficient of *any* $H_p(x)$ cannot be obtained from the HDE. The above choice makes the leading coefficient of $H_p(x)$ to be 2^p.) Thus the differential recurrence relation is [1,2,4]:

$$DH_p(x) = 2p\ H_{p-1}(x),\ p\geq 1. \tag{6}$$

From (1), we have $[D^2 - 2\ x\ D + 2(N+1)]\ H_{N+1}(x)=0$. Eliminating the derivatives using (6), we arrive at the three-term recurrence relation satisfied by the Hermite polynomials [1,2,4] :

$$H_{N+1}(x) = 2\,x\,H_N(x) - 2\,N\,H_{N-1}(x),\ N \geq 1. \qquad (7)$$

Using (4), we have $H_{2m+1}(0) = 0$. One may be tempted to ask whether $H_{2m}(0)$ can also follow suit. We show that the answer is definitely in the negative.

SIGNIFICANCE OF $H_{2m}(0) \neq 0$ AS SEEN THROUGH THE WINDOW OF THE HDE

The HDE for even degree polynomials is

$$(D^2 - 2\,x\,D + 4m)\,H_{2m}(x) = 0,\ m \geq 0. \qquad (8)$$

Since $H_{2m}(x)$ is an even polynomial in x (see (4)), its derivative with respect to x is an odd polynomial in x and hence vanishes at x = 0. If the constant term of $H_{2m}(x)$ were zero, then the solution of (8) and its derivative would be zero at the same point x = 0. As an initial value problem in the theory of second order linear differential equations has a unique solution [5,6], the solution of (8) would be identically zero. As we are interested in having a non-trivial polynomial solution of the HDE, we must choose $H_{2m}(0)$ to be non-zero.

What we give below is a "constructive" proof of the fact that $H_{2m}(x)$ is non-trivial, with a non-zero leading term. This is an alternative to the one generally given in books [1,6] for finding the polynomial solution from the HDE. Differentiating both sides of (8) with respect to x successively 2k times, using the Leibnitz theorem, we have

$$D^{2k+2}H_{2m}(x)|_{x=0} = 4(k - m)\,D^{2k}\,H_{2m}(x)|_{x=0}. \qquad (9)$$

As $H_{2m}(0)$ is non-zero, $m \geq 1$, iteration of (9) shows that $D^{2s}\ H_{2m}(x)$ has to be non-zero at $x = 0$ when $0 \leq s \leq m$. If

$$H_{2m}(x) = \sum_{s=0}^{m} A_{2s}\, x^{2s}, \tag{10}$$

then

$$A_{2s} = (-1)^s\, 4^s\, m!\, H_{2m}(0)/[(2s)!\,(m-s)!] \neq 0. \tag{11}$$

Using (1), (4) and (6), we now show that $H_{2m}(0) \neq 0$ will also make $H_{2m+1}(x)$ a non-trivial polynomial. The Hermite polynomial of odd degree is an odd polynomial and hence is zero at the origin. The differential equation satisfied by it implies that it is not identically zero since its derivative is different from zero at the origin: $DH_{2m+1}(x)|_{x=0} = 2(2m+1)H_{2m}(0) \neq 0$.

It follows from the differential recurrence relation and (8) that

$$4(2m+2)\,(2m+1)\,H_{2m}(0) = D^2\,H_{2m+2}(x)|_{x=0}$$

$$= -4\,(m+1)\,H_{2m+2}(0). \tag{12}$$

Hence $H_{2m+2}(0) \neq 0$, if $H_{2m}(0)$ is non-zero for a particular m. Iteration of (12) now yields that $H_{2m}(0) \neq 0$, for all $m \geq 0$. Since every non-zero $H_{2m}(0)$ saves both $H_{2m}(x)$ and $H_{2m+1}(x)$ from being identically zero, it is clear that a single non-zero $H_{2m}(0)$ can save the entire set of Hermite polynomials.

THROUGH THE THREE-TERM RECURRENCE RELATION AND ZEROS

From the three-term recurrence relation, (7), we have

$$H_{2m}(0) = -2(2m-1)\, H_{2m-2}(0), \; m \geq 1 . \qquad (13)$$

As every polynomial belonging to a set of simple polynomials has a precise degree with a non-zero leading term [4,11], $H_0(0)$ cannot be zero if we demand the Hermite polynomials to be simple polynomials. From (13) it follows that $H_{2m}(0) \neq 0$, $m \geq 0$.

If the constant term of $H_{2r}(x)$ were zero for some $r \geq 1$, then $H_{2r}(x) = x^2 \, [A_{2r} \, x^{2r-2} + \ldots + A_2]$. (Here we use only the fact that $H_{2r}(x)$ has a definite parity (see (4)) and hence it contains only even powers of x.) Therefore, $H_{2r}(x)$ would have a zero of multiplicity 2 at $x = 0$, contradicting the fact that the zeros of real orthogonal polynomials are distinct [4,11]. (This result is a consequence of the orthogonal property [4,11] which itself is a natural consequence of the Sturm-Liouville form of the HDE [1,3,5]. See Section 6.) Hence $H_{2r}(0) \neq 0$, for all $r \geq 0$.

OTHER WINDOWS

Readers may wonder why we have not looked either through the window of the generating function [1,2,4]

$$\exp(2xt - t^2) = \sum_{m=0}^{\infty} H_m(x)\, t^m/m!, \qquad (14)$$

or through that of the Rodrigues' representation [1,2,4,

$$H_n(x) = (-1)^n \exp(x^2) D^n(\exp(-x^2)), \; n \geq 0, \quad (15)$$

or through that of the simple but beautiful relation [1,7,8,9]

$$H_n(x) = (2x - D)^n 1, \; n \geq 0. \quad (16)$$

Each one of the above three relations has been proved to be a starting point for studying the properties of the Hermite polynomials. For proofs, see Refs. [2], [7], and [8,9], respectively. Like the power series expansion [1,2,4] of $H_n(x)$, they specify the Hermite polynomials completely, leaving no room for any speculation whether $H_{2n}(0)$, $n \geq 0$, can be zero or not.

STRUM-LIOUVILLE FORM AND SZEGÖ'S RESULT

The Strum-Liouville form of the HDE is [1,3,5]

$$[L + 2p \exp(-x^2)] \, H_p(x) = 0, \quad p \geq 0, \quad (17)$$

where the Strum-Liouville operator L is D $\exp(-x^2)$D. As the eigenfunctions of the Sturm-Liouville operator belonging to distinct eigenvalues are orthogonal with respect to the weight function [1,3,5],

$$\int_{-\infty}^{+\infty} \exp(-x^2) \, H_m(x) \, H_n(x) \, dx = 0, \quad m \neq n. \quad (18)$$

The even nature of the associated weight function and the symmetry of the interval of orthogonality with

respect to the origin demand, according to an elegent and general result from Szegö [11] that $H_n(x)$ *can contain* only those powers of x which are congruent to n(mod 2), i.e., it is an odd or even polynomial according to whether n is odd or even. Using Szegö's result [11], the properties of the zeros [4,11] of $H_n(x)$ and Descartes' rule of signs, it has been shown [10] that $H_n(x)$ *does contain* only those and all those powers of x which are congruent to n(mod 2). Hence in

$$H_{2n}(x) = \sum_{m=0}^{n} A_{2m}\, x^{2m}, \; n \geq 0, \tag{19}$$

$$H_{2n+1}(x) = x \sum_{s=0}^{n} A_{2s+1}\, x^{2s}, \; n \geq 0, \tag{20}$$

of the coefficients A_{2m} and A_{2s+1} can be zero. As simple corollaries, we have,

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) \neq 0, \quad n \geq 0 \tag{21}$$

With $H_0(0) = 1$, it follows from (12) or (13) and iteration that

$$H_{2n}(0) = (-1)^n\, (2n)!/n! \neq 0, \quad n \geq 0. \tag{22}$$

CONCLUSIONS

What a tremendous potential the particularly pleasant property $H_{2n}(0) \neq 0$ has! As $H_{2n}(0)$ 0, $H_N(x) \not\equiv 0$, $N \geq 0$. One member of a family saves the entire family: The constant term of a Hermite polynomial of even degree is the saviour of the Hermite

polynomials. Our analysis throws light on the usefulness of the Wronskian in obtaining the parity and recurrence relations satisfied by the Hermite polynomials. While it is good to know a better proof of a result, it is better to know good proofs of the same also. One may extend our approach to other real orthogonal polynomials associated with an even weight function and an interval of orthogonality symmetric with respect to the origin (e.g. Legendre polynomials and Chebyshev polynomials [1,2]).

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A NOTE ON THE MODULAR GROUP RING OF THE SYMMETRIC GROUP S_n

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In [2] the author has proved if G is a group of order two and K is a field with more than two elements then G^*, the mod p-envelope of G has a semi-group structure with a non trivial idempotent in it. It is interesting to note when G is the symmetric group S_n and K is a field having two elements 0 and 1 then G^* the mod p-envelope of G has a semi-group structure with non trivial idempotents in it and the number of elements in G^* is $2^{\lfloor n-1}$. For the sake of completeness we prove the following propositions. For definition's please refer [1] and [2].

Proposition 1. Let G be the symmetric group of permutations of (1,2,3) and $K=(0,1)$ be the field of two elements. Then G^* is a semi-group with non trivial idempotents and the number of elements in G^* is $32 = 2^5 = 2^{\lfloor 3-1}$.

Proof. $G = \{ \binom{1\,2\,3}{1\,2\,3} = 1, \binom{1\,2\,3}{1\,3\,2} = p_1, \binom{1\,2\,3}{3\,2\,1} = p_2, \binom{1\,2\,3}{2\,1\,3} = p_3,$

$\binom{1\,2\,3}{2\,3\,1} = p_4, \binom{1\,2\,3}{3\,1\,2} = p_5 \}$.

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$$G^* = \{1+U\}$$

$$= \{1, p_1, p_2, p_3, p_4, p_5, 1+p_1+p_2, 1+p_2+p_3, 1+p_1+p_3, 1+p_1+p_4, 1+p_1+p_5, 1+p_2+p_4, 1+p_2+p_5, 1+p_3+p_4, 1+p_3+p_5, 1+p_4+p_5, p_1+p_2+p_3, p_1+p_2+p_4, p_1+p_2+p_5, p_1+p_3+p_4, p_1+p_3+p_5, p_1+p_4+p_5, p_2+p_4+p_5, p_3+p_4+p_5, p_2+p_3+p_4, p_2+p_3+p_5, 1+p_1+p_2+p_3+p_4, 1+p_1+p_2+p_3+p_5, 1+p_1+p_2+p_4+p_5, 1+p_2+p_3+p_4+p_5, 1+p_1+p_3+p_4+p_5, p_1+p_2+p_4+p_5+p_3\}.$$

Clearly G^* is a semi-group with identity

$$\{1+p_4+p_5, p_1+p_2+p_4, p_1+p_2+p_5, p_1+p_3+p_4, p_1+p_3+p_5, p_2+p_3+p_3, p_2+p_3+p_5, 1+p_1+p_2+p_4+p_5, 1+p_1+p_3+p_4+p_5 \text{ and } p_1+p_2+p_3+p_4+p_5\}$$

are nontrivial idempotents of G^*.

Proposition 2. Let G be the symmetric group of permutations of (1,2,3,4) and K = (0,1) be the field of two elements. Then G^* is a semi-group with non trivial idempotents and the number elements in G^* is $2^{23} = 2^{\lfloor 4 - 1}$.

Proof. $G^* = \{1+U\}$

$$= \{\alpha = \Sigma\alpha_i g_i \;/\; \Sigma\alpha_i = 1, \; g_i \in G, \; \alpha_i \in (0,1)\}$$

$$|G^*| = 2[24C_1+24C_3+24C_5+24C_7+24C_9+24C_{11}]$$

$$= 2^{23} \text{ (I have verified by calculations).}$$

Consider

$$\alpha_1 = 1 + \binom{1\,2\,3\,4}{3\,2\,4\,1} + \binom{1\,2\,3\,4}{4\,2\,1\,3}$$

$$\alpha_2 = 1 + \binom{1\,2\,3\,4}{2\,3\,1\,4} + \binom{1\,2\,3\,4}{3\,1\,2\,4} \text{ and}$$

$$\alpha_3 = 1 + \binom{1\,2\,3\,4}{2\,4\,3\,1} + \binom{1\,2\,3\,4}{4\,1\,3\,2}$$

are some of the non-trivial idempotents in G^*.

Promblem. Derive a method by which we can enumerate the number of non-trivial idempotents in G^*.

Theorem 3. Let G be the symmetric group of degree n. Let $K = (0,1)$ be the field of two elements, G^* the mod p envelope of G has $2^{\lfloor n-1}$ elements and it has non-trivial idempotents and forms a semigroup.

Proof. $G^* = \{1+U\}$

$$= \lfloor nC_1 + \lfloor nC_3 + \lfloor nC_5 + \ldots + \lfloor nC_{\lfloor n-1}$$

$$= 2^{\lfloor n-1}.$$

Now every element in G^* is such that the sum of the coefficients is 1. This forces the number of terms of every element in G^* is odd. Clearly G^* is closed for the product of odd terms modulo two is also odd. Hence G^* is semi-group. $1 \in G^*$ so G^* is also a semi-group with identity. G^* has non-trivial idempotents for take.

$$\alpha = 1 + \binom{1\,2\,3\,4\ldots n}{2\,3\,1\,4\ldots n} + \binom{1\,2\,3\,4\ldots n}{3\,1\,2\,4\ldots n}.$$

Clearly $\alpha^2 = \alpha$ hence $\alpha \in G^*$ is a non trivial idempotent.

Example 1. Let $S_2 = \{ \binom{1\,2}{1\,2}, \binom{1\,2}{2\,1} \}$, $K = (0,1,2)$ be the field of three elements. What is G^*?

Proof. $G^* = \{1+U\}$
$= \{1, \binom{1\,2}{2\,1}, 2+2\binom{1\,2}{2\,1}\}$.

Order of $G^* = 3^{\lfloor 2-1} = 3$ and $2+2\binom{1\,2}{2\,1}$ is the non-trivial idempotent of G^*.

Example 1 can be extended to any field of p-elements

p a prime $G = \{\binom{1\,2}{1\,2}, \binom{1\,2}{2\,1}\}$ and let

$$K = (0,1,2,\ldots,p-1)$$

$$G^* = \{1, (p-1)\binom{1\,2}{2\,1}+2, 2\binom{1\,2}{2\,1}+p-1, (p-2)\binom{1\,2}{2\,1}+3, 3\binom{1\,2}{2\,1}+p-2, \ldots, \frac{p+1}{2}+\frac{p+1}{2}\binom{1\,2}{2\,1}\}.$$

G^* is a semi-group with p-elements, and $\frac{p+1}{2}+\frac{p+1}{2}\binom{1\,2}{2\,1}$ is an idempotent of G^*. Then order of

$$G^* = p^{\lfloor 2-1} = p.$$

Hence we generalize this as a problem.

Problem. If G is symmetric group of order n, $K = (0,1,2,\ldots,p-1)$ be a semi-group of order $p^{\lfloor n-1}$ and has non-trivial idempotents.

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AN ALTERNATIVE APPROACH TO TOLLMIEN'S ENERGY SPECTRA DERIVED FROM HEISENBERG'S EQUATION OF TURBULENT FLOW

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ABSTRACT

Tollmien (1952-53) had to work with experimenental data of windtunnel measurements due to Stewart and Townsend (1951) for the evaluation of turbulent energy spectra. In this paper similar categories of spectra have been shown to be alternatively derivable from Sen's (1951) self preserving solution of Heisenberg's (1948) equation and by extending numerical tables of Ghosh (1955) the author has ultimately obtained self-similar spectrum curves somewhat analogous to the normalized apectra calculated by Tollmien at different times of turbuleNt fluid flow.

1. INTRODUCTION

It is well known that experimental studies of turbulence are based on relevant data obtained experimentally at least at a distance of 20 mesh length behind the grids of wind tunnel. Tollmien's numerical evaluation of turbulent energy spectra (1952-53)at times t=0 t=0.2, t=0.5, t=1 has a footing on experimental results at 30mesh length. If F(K,t) be the energy spectrum function of homogeneous- isotropic turbulence for wave number K at time t, the decay of turbulence occurs according to Heisenberg's (1948) equation

$$-\frac{\partial}{\partial t}\int_{\infty}^{k} F(k,t)dk=2(\nu+n_k)\int_{\infty}^{k} k^2 F(k,t)dk \qquad \ldots (1)$$

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where ν is kinematic viscosity and η_k is the turbulent viscosity given by

$$\eta_k = \nu_H \int_k^{\infty} \frac{d\ F(K,t)}{k^3} dk$$

ν_H being a constant.

Since the integro-differential equation (1) contains only the first order time derivative, is follows that its solution is uniquely determined by imposing initial condition of the form

$$F(K,t)\Big|_{t\to 0} = F(K,0) = F_o(K),\ (\text{say}).$$

Therefore, if the form of the spectrum at initial time is known from experiment, one can determine the spectrum for all subsequent times by numerical integration Thist type of calculation has been incidentally carried out by Tollmien for Heisenberg's equation (1) and the data of Stewart and Townsend(1951). Tollmien took $F_o(K)$ in the form of data of Fig.(a) and Y_H=0.45 according to Proudman's estimate (1951). Furthermore, Tollmien attempted to normalize his solution in accordance with the data of Fig.(a) so that the values of F(K,t) for large K were as near time-independent as possible. The curves obtained in this way vide Fig.(b) were found to be qualitattively similar to empirical curves of Stewart and Townsend shown in Fig. (a). Although it has been stated that the resulting function F(K,t) did not display self preserving behaviour in time, yet the current evaluation will lead to inference otherwise if we start with Ghosh's (1955) numerical tables.

Nevertheless, turbulent energy calculated by Tollmien was found to be in very good agreement with the empirical "minus one law" right upto t=0.5 and this law was usually regarded as an evidence for self-preservation.

In tins context it may be mentioned that Karman-Howarth(1938) defined self preserving solutions as to be those whose curves do not undergo a change of shape with time, the only change being the change of scale. The motion in a sense remains similar to itself.

Heisenberg has shown that the equation (1) admits of a self similar solution of the form for

$$F(k,t) = \frac{1}{\sqrt{t}}\, f(K\sqrt{t}) \qquad \ldots (2)$$

for which f(x) with x=kt obeys an ordinary differential equation derivable from (1). This set of self-similar solutions was numerically calculated by Chandrashekhar (1949). Such numerical solutions so obtained on the basis of equation (2) have obvious break of similarity at the left end of self similar curves [cf. Karman & Lin (1951)].

In order to overcome this drawbhack Sen(1951) suggested a bunch of solutions of Heisenberg's equation (1) in the form

$$F(K,t) = \frac{S^3}{\nu^2 k_o^3 t_o^3 \tau^2}\, f\left(S\frac{K}{K_o}\right) \qquad \ldots (3)$$

where $s=s(\tau)$, $T=t/t_o$ and K_o, t_o are constant. Using Reynolds number R as

$$\frac{1}{R} = \nu k_o^2 t_o \text{ and } SK/K_o = x,$$

the equation of f(x) is obtined by substitution of (3) in (1) as

$$2(1-c)\int_o^x f(x)dx - cxf(x) = 2\int_x^\infty \sqrt{\frac{f(x)}{x^3}}\,dx \int_o^x x^2 f(x)dx + \frac{2\tau}{S^2R} \qquad \ldots (4)$$

where c is parameter introduced by Sen as $c = \frac{1}{s}\frac{ds}{dt}$ i.e. $s = \alpha\tau^c$, α being a constant.

Neglecting the last term of equation (4) for early period of decay process where Reynolds number of fluid flow is infinitely large, Ghosh has obtained numerical tables by means of Sen's self similar solutions.

$$F(k,t) = \left[\frac{1}{\nu_H^2 k_o^3 t_o^2} \left(\frac{t}{t_o}\right)^{3c-2} f\left(\frac{k}{k_o}\left\{\frac{t}{t_o}\right\}\right)\right]_{c=\frac{2}{7}}$$

with asymtotic behaviours

$$f(x) \approx x^4, (x \to 0) \text{ and } f(x) \approx x^{-5/3}, (x \to \infty).$$

In doing so Ghosh also followed Chandrasekhar for introduction of two auxiliary functions as

$$g = x^3 f(x) \text{ and } y(x) = \int_0^x x^2 f(x)dx$$

to recast the equation (4) for $R \to \infty$ into the form

$$cg^{3/2}g'' + (4+g')y + 2g^{1/2}(2-cg') - cg) - 4g = 0 \qquad \dots (5)$$

where $g' = \frac{dg}{dy}$.

If $c=\frac{1}{2}$, the relation (5) gives Chandrasekhar's equation for the preparation of numerical table of $(y, g(y))$ and thereafter $(x, f(x))$, whereas $c = \frac{2}{7}$ leads to Ghosh's tables for $(y, g(y))$, and consequently $(x, f(x))$. It is observed that desipite $(x, f(x))$ curves of Chandrasekhar's tabular values and those due to Ghosh are by and large similar in pattern, yet it is obvious that there is a break of similarity at the left end portion of each individual $(x, f(x))$ curves due to Chandrasekhar's tables. This is because of the fact that the asymptotic behaviours of Chandrasekhar's tables involve $f(x) \approx x$, $(x \to o)$ and $f(x) \approx x^{-5/3}$, $(x \to \infty)$ of which, the first one i.e. $f(x) \approx x, (x \to 0)$ leads to break of similarity while $F(K,t)|_{t \to o}$ is to be accounted for through a behaviour of $f(x)$ at $x \to 0$. Hence to proceed with this form for $f(x) = f\{\frac{k}{k_o} \sqrt{\frac{t}{t_o}}\}$ as $t \to 0$ is anomalous.

As such in the present paper it has been attempted to generate numerical values of F(K,t) when $t \to 0$ as obtainable from Sen's self-similar solution of equation (1) for parametric value $c=\frac{2}{7}$ by making an extension of Ghosh's numerical results transformed into new tables that eventually yield identical curves resembling Tollmien's spectra and by dint of this approach it became possible to maintain the self preserving behaviour throughout the entire specturm as well.

2. APPROACH TO THE POBLEM

In the present problem the equation (1) has, in fine, the more general solution of the from

$$F(k,t) \approx \text{const.}\, t^{3c-2} f(kt^{c})$$

where c is an arbitarary constant.

For $c=\frac{2}{7}$, $(0<c<\frac{2}{3})$ we have in particular

$$F(k,t) \approx \frac{1}{\nu^2_H K^3_o t^2_o} \left(\frac{t}{t_o}\right)^{-8/7} f\left(\frac{k}{k_o}\left\{\frac{t}{t_o}\right\}^{2/7}\right) \quad \ldots (6)$$

Evidently asymptotic behaviours of F(k,t) are here found to be

$$F(k,t) \approx k^4, (k\to 0) \quad F(k,t)=k^{-5/3}, (k\to\infty)$$

The first one of the asymptotes was supported in the works of C.C. Lim (1947) and G.K. Batchelor (1949), while the other asymptotic behaviour leads to Kolmogorov (1941) spectrum. In addition to theoretical background, in fact the spectrum K^4, $(k\to 0)$ has got also strong experimental confirmation according to Debnath (1978) and the evaluation of F(k,t) for $c = \frac{2}{7}$ concerns in this case the derivation of numerical solutions of (x,f(x)) without any break of

self-similarity at $x \to o$ i.e. $t \to o$ or $k \to o$ as well as $t \to o$. According to our requirement we are to set forth the intial conditions as to be

$$F(k,t)\big|_{t\to o} = \frac{1}{\nu_H^2 K_o^3 t_o^2}\left(\frac{t}{t_o}\right)^{-8/7} [f(x)]_{x\to o} \text{ for } x=\frac{k}{k_o}\left(\frac{t}{t_o}\right)^{2/7} \quad \ldots (7)$$

where $(x, f(x))$ data are to be collected from numerical tables of Ghosh and K_o, t_o being unity for the present instance. The reliability of new set of results may be verified by the condition $F(k,t)=\Lambda k^4$ as $k \to o$, where Λ is Loitsiansky's (1939) invariant in respect of the tables concerned

3. NUMERICAL TABLES AND GRAPHS

In the light of aforesaid facts we have developed the following numerical tables I & II where first, second columns represent data taken from Ghosh's tables and for $x=\frac{k}{k_o}\left(\frac{t}{t_o}\right)^{2/7}$ the numerical values of third, fourth, fifth columns of each table have been calculated at t=0.5, 0.2, 0.1 respectively. On the basis of such extended tables we have generated turbulent energy spectrum converted finally in terms of curves (Fig.1 and Fig.2) which are found to be akin to the Tollmien's spectra and thereby leading to remarkable modification in the self preserving nature of spectra as pictured by the current approach.

CONCLUDING REMARKS

(i) Thus to start with fourth power law $f(x) \approx x^4, (x \to 0)$ meaning $x = \frac{k}{k_o} (\frac{t}{t_o})^{2/7} \to 0$ includes the behaviour at $t \to 0$ and hence having regard to Sen's (1951) self similar solution of Heisenberg's (1948) energy decay equation one can safely work with the tables of Ghosh (1955) in lieu of wind-tunnel data of Stewart and Townsend (1951) for the evaluation of turbulent energy spectrum at different times where the resulting function F(k,t) exhibits self preserving behaviour with time as evident from the graph of the present note.

(ii) One may extrapolate freely at the end of the tables extended in the current work as there is no break of self similarity anywhere in the whole range of the spectra and such extrapolation at the extreme upper end of the tables concerns the evaluation of Loitsianky's constant Λ that remains independent of time.

(iii) At the initial stage of decay phenomean when low frequency part of the spectrum plays significant role or in otherwords when the low frequency eddies are predominant, then variety of spectra satisfying similarity law may be possible depending on the method of production of turbulence as the parameter c being proposed to be associated with the mode of excitation of turbulence and this approach is valid at a stage of turbulence of negligible viscous dissipation by the wave numbers concerned so that the inertia terms of Navier-stokes equation are mainly responsible for the entire transfer of energy down the spectrum of such self preserving isotropic turbulence.

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TABLE : 1

x	f(x)	k	k	k
0.3737	0.1314	0.4555	0.5919	0.7215
0.4166	0.1660	0.5078	0.6598	0.8043
0.4708	0.2067	0.5739	0.7456	0.9090
0.4910	0.2203	0.5985	0.7777	0.9480
0.5320	0.2442	06485	0.8426	1.0271
0.5656	0.2592	0.6895	0.8958	1.0920
0.6623	0.2792	0.8074	1.0490	1.2787
0.7360	0.2720	0.8972	1.1657	1.4210
0.8006	0.2525	0.9759	1.2680	1.5457
0.8607	0.2279	1.0492	1.3632	1.6618
0.9200	0.1979	1.1215	1.4571	1.7762
0.9810	0.1664	1.1959	1.5537	1.8940
1.0466	0.1355	1.2758	1.6576	2.0207
1.1206	0.1003	1.3660	1.7748	2.1635
1.2111	0.0668	1.4763	1.9182	2.3383
1.8446	0.0037	2.2486	2.9215	3.5614

TABLE : II

x	f(x)	k	k	k
0.3736	0.1320	0.4554	0.5917	0.7213
0.4160	0.1695	0.5071	0.6589	0.8032
0.4687	0.2158	0.5714	0.7423	0.9049
0.4882	0.2323	0.5951	0.7732	0.9426
0.5273	0.2628	0.6428	0.8351	1.0181
0.5582	0.2845	0.6805	0.8841	1.0777
0.6469	0.3295	0.7886	1.0246	1.2490
0.7109	0.3452	0.8666	1.1259	1.3725
0.7638	0.3481	0.9311	1.2097	1.4747
0.8104	0.3436	0.9879	1.2835	1.5646
0.8530	0.3346	1.0398	1.3510	1.6469
0.9310	0.3083	1.1349	1.4745	1.7975
1.0042	0.2757	1.2241	1.5905	1.9388
1.0759	0.2397	1.3115	1.7040	2.0772
1.1492	0.2019	1.4009	1.8201	2.2186
1.2270	0.1633	1.4957	1.9433	2.3690
1.3137	0.1245	1.6014	2.0807	2.5364
1.4168	0.0861	1.7271	2.2440	2.7354
1.5547	0.0490	1.8952	2.4624	3.0016
1.8078	0.0147	2.2037	2.8632	3.4903
2.3858	0.0009	2.9083	3.7787	4.6063

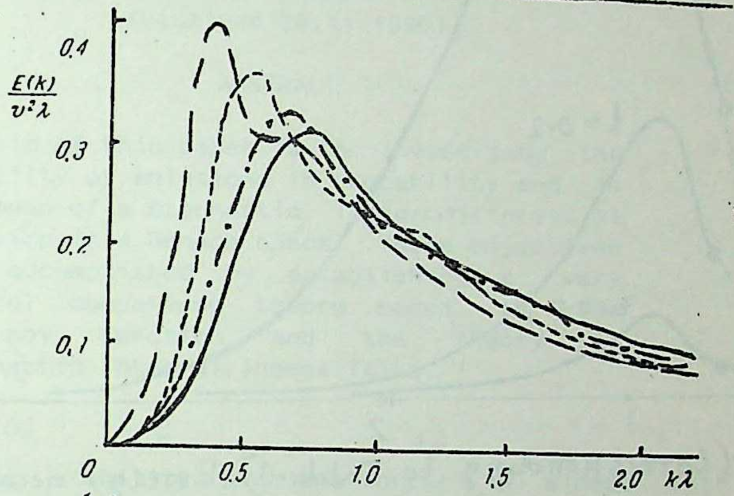

FIG. (a) Normalized spectral density functions at different distances from the grid [Stewart and Townsend (1951)].

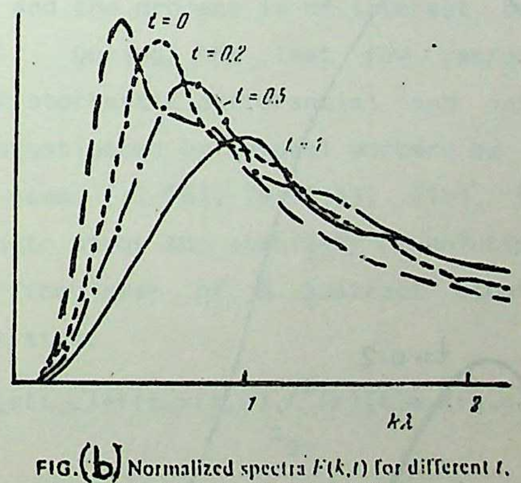

FIG. (b) Normalized spectra $F(k,t)$ for different t, calculated by Tollmien (1952 - 1953).

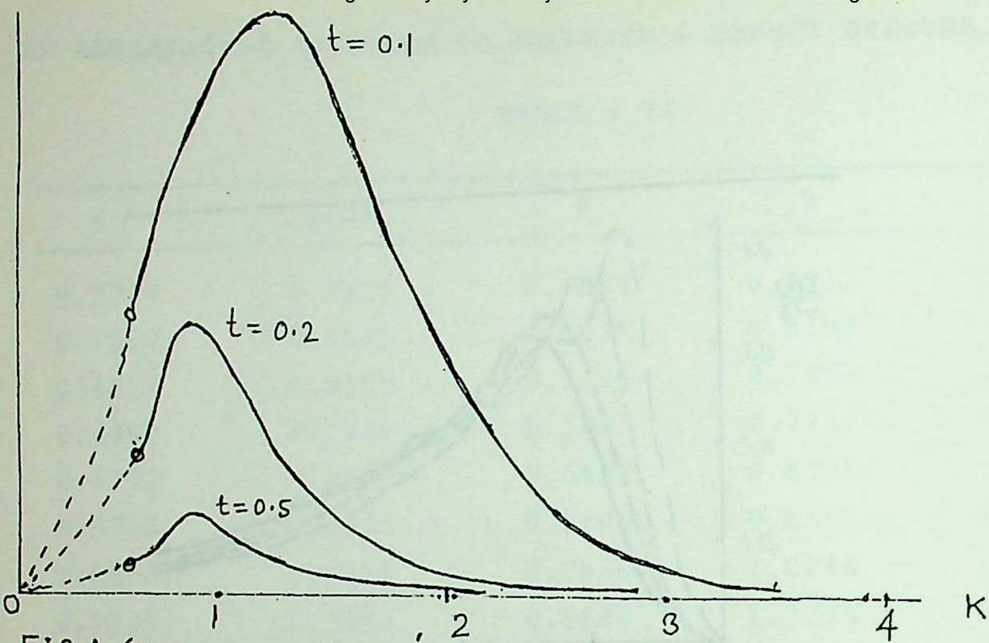

FIG.1 (Corresponding to Table-I) Energy decay spectra F(K,t).

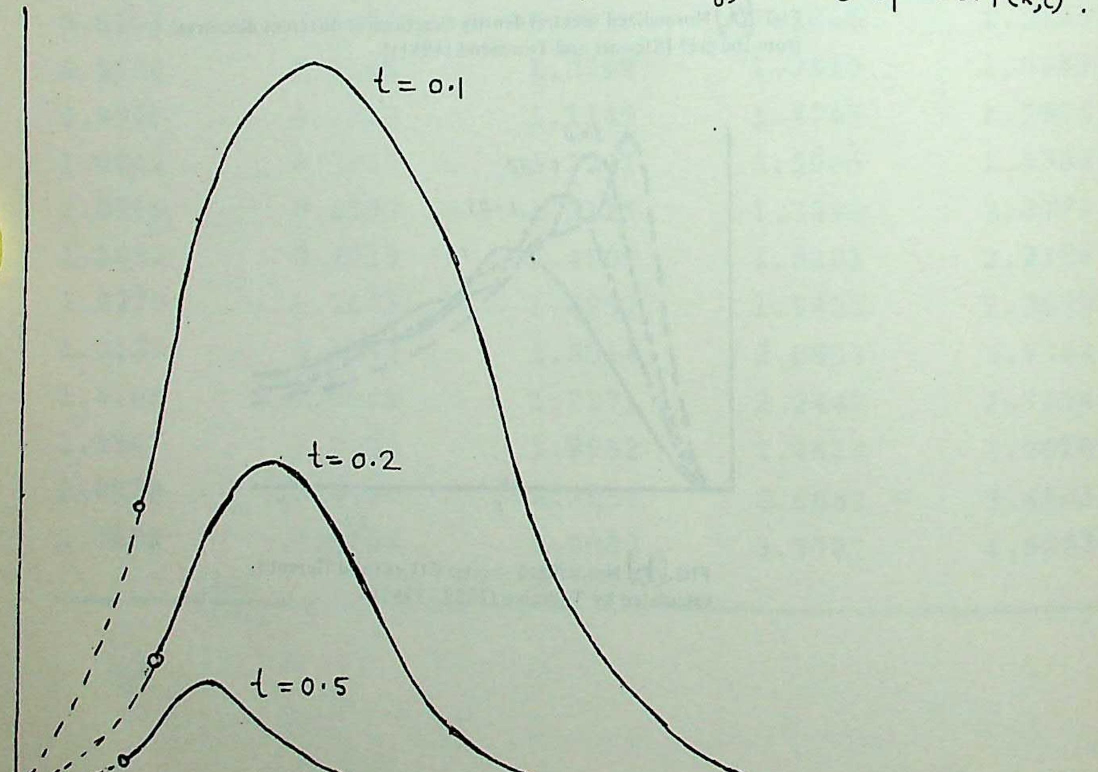

FIG.2 (Corresponding to Table-II) Energy decay spectra F(K,t).

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ON THE STABILITY OF A STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS IN A BANACH SPACE

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ABSTRACT

The aim of this paper is to investigate the stability of solutions in probability and in the mean of a stochastic integrodifferential equation in a Banach space. These objectives are accomplished by establishing a very general comparison theorm based on the Lyapunov function and the theory of stochastic integral inequalities.

1. INTRODUCTION

Stochastic stability problems occur in almost all phases of physics, control theory, numerical analysis and economics where dynamical models subject to random disturbances appear, and the process is of interest over a long period of time. During the last few years, the stability analysis of stochastic differential and integral equations has been investigated by several workers by using different techniques (see, [1]-[5], [8]-[12], [14], [15]). In this paper we wish to study the stability of solutions in probability and in the mean of a abstract stochastic integrodifferential system.

$$(1)\quad x'(t,\omega) = A(t,\omega)x(t,\omega)+f(t,x(t,\omega),\int_{t_0}^{t}(K)[t,s,x(s,\omega),\omega]ds,\omega)$$

$$+g(t,x(t,\omega),\omega),\quad x(t_0,\omega) = x_0(\omega),$$

given certain connective information concerning the solutions of the corresponding abstract stochastic integrodifferential system

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$$(2)\quad y'(t,\omega) = A(t,\omega)y(t,\omega)+f[t,y(t,\omega), \int_{t_0}^{t} K[t,s,y(s,\omega),\omega]ds,\omega],$$

$$y(t_0,\omega) = x_0\ (\omega),$$

and the auxiliary stochastic integrodifferential equation

$$(3)\quad r'(t,\omega)= H_1(t,r(t,\omega),\int_{t_0}^{t} K_0[t,s,r(s,\omega),\omega]ds,\omega)+H_2(t,r(t,\omega),\omega),$$

$$r(t_0\omega) = r_0(\omega),$$

where

(i) $\omega \in \Omega$ the supporting set of a complete probability measure space (Ω,F,P);

(ii) $x(t,\omega)$, $y(t,\omega)$ are the unknown random processes which map $R^+ \times \Omega$ into X, where X is a seperable Banach space with norm $\| \, . \, \|$;

(iii) for each $t \in R^+$, the set $A(t,\omega)$ is a family of random operators [1] in X, with domain $D[A(t,\omega)]$;

(iv) $g(t,x,\omega)$, $K[t,s,x,\omega]$ and $f(t,x,\bar{x},\omega)$ are Borel measurable maps from $R^+ \times X \times \Omega$, $R^+ \times R^+ \times X \times \Omega$ and $R^+ \times X \times X \times \Omega$ into

X respectively such that $g(t,0,\omega) = 0$, $f(t,0, \int_{t_0}^{t} (K)$

$[t,s,0,\omega]ds,\omega) = 0$;

(v) $H_2(t,r,\omega)$, $K_0[t,s,r,\omega]$ and $H_1(t,r,\bar{r},\omega)$ are Borel measurable maps from $R^+ \times R \times \Omega$, $R^+ \times R^+ R \times \Omega$ and $R^+ \times R \times R \times \Omega$ into R respectively such that $H_2(t,0,\omega) = 0$, $H_1(t,0,\int K_0[t,s,0,\omega], \omega) = 0$.

Recently, by assuming the existence of solutions of stochastic differential and integral equations Kats and Krasovskii [2], Martinjuk [8], Ladde [4], Ladde, Lakshmikantham and Liu [5], Rao and Manougian [14], Pachpatte [10]-[12] and others have studied the stability and boundedness of solutions by employing the second method of Lyapunov and the theory of differential and integral inequalities. Here, by assuming the existence of global mean square continuous solutions of (1), (2) and (3) for all $t \geq t_0$ and employing Lyapunov functions and the theory of integral inequalities we give sufficient conditions for the stability in the probability and in the mean of the solutions of (1) by means of comparison with the connective stablility in the probability and in the mean of (2) and (3).

2. PRELIMINARIES

In our subsequent discussion, we are interested in random linear operators difined on subsets of $X \times \Omega$. Suppose that for each $t \in R^+$, $A(t,\omega)$ is a closed random linear operator such that $[I-\alpha A(t,\omega)]^{-1}$ exists for a real α, where I is the identity operator. Suppose further that $D[I-\alpha A(t,\omega)]^{-1}$ is dense in X. Denote the resolvant of $A(t,\omega)$ by $R(\alpha, A(t,\omega)) \equiv [I\alpha A(t,\omega)]^{-1}$. Then the following results are known [1]:

(i) $R(\alpha,A(t,\omega))$ is a random linear operator for each $t \in R^+$;

(ii) $D[R(\alpha,A(t,\omega))] = X$;

(iii) $[I-\alpha A(t,\omega)]\, R(\alpha,A(t,\omega))x = x$ for $x \in X$;

(iv) $R(\alpha,A(t,\omega))\, [I-\alpha A(t,\omega)]x = x$ for $x \in D\,[A(t,\omega)]$.

We shall use the following definitions in our subsequent discussion, (see,[4, 5, 9, 12]).

Definition 1. Let $x(t,\omega) \in D[A(t,\omega)]$ be a random variable with values in X for each $t \in R^+$. Then $x(t,\omega)$ is called a solution of (1) if $x(t,\omega)$ has a strong derivative in X and satisfies (1).

Definition 2. The trivial solution of (1) is said to be (SP_1) *Stable in probability*, if for each $\varepsilon > 0$, $\eta > 0$, $t_0 \in R^+$, there exists $\delta = \delta(t_0,\varepsilon,\eta) > 0$ such that

$$P\{\omega : \|x_0(\omega)\| > \delta\} < \eta$$

implies

$$P\{\omega: \|x(t,\omega)\| \geq \varepsilon\} < \eta, \ t \geq t_0;$$

(SP_2) *asymptotically stable in probability*, if it is stable in probability and, if for any $\varepsilon > 0$, $\eta>0$, $t_0 \in R^+$, there exist positive number $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\varepsilon\eta)$ such that

$$P\{\omega : \|x_0(\omega)\| > \delta_0\} < \eta$$

implies

$$P\{\omega : \|x(t,\omega)\| \geq \varepsilon\} < \eta, \ t \geq t_0 + T;$$

(SM_1) *Stable in the mean*, if for each $\varepsilon>0$, $t_0 \in R^+$, there exists a $\delta = \delta(t_0,\in) > 0$ such that the inequality

$$E[\|x_0(\omega)\|] \leq \delta$$

implies

$$E[\|x(t,\omega)\|] < \varepsilon, \ t \geq t_0;$$

(SM_2) *asymptotically stable in probability*, if it is stable in the mean and, if for any $\varepsilon > 0$, $t_0 \in R^+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T=T(t_0,\varepsilon)$ such that the inequality

$$E[\|x_0(\omega)\|] \leq \delta_0$$

implies

$$E[||x(t,\omega)||] < \varepsilon, \ t \geq t_0+T.$$

Based on Definition 2, one can formulate other difinitions of stability and boundedness (see, [4,5,14]) analogously.

Defination 3. A function b (r) is said to belong to the class K if $b \in C[R^+ + R^+]$, $b(0) = 0$, $b(r)$ is strictly increasing in r.

Definition 4. A function b(r) is said to belong to the class VK, if $b \in C\ [R^+, R^+]$, $b(0) = 0$, $b(r)$ is a convex and strictly increasing in r.

Definition 5. A function a(t,r) is said to belong to the class CK if $a \in C(R^+ X R^+, R^+]$, $a(t,0) = 0$, and $a(t,r)$ is concave and increasing in r for each fixed $t \in R^+$.

In order to avoid monotonicity, hereafter, it will be understood, unless otherwise specified, that the functions involved in (1)-(3) satisfy the conditions (iv) and (v) given in Section 1, and furthermore all equalities, inequalities, and relations that involve the random processes will hold with probability 1.

3. COMPARISON THEOREM

In this section, we develop a very general comparison theorem for stochastic integrodifferential system (1) which plays an important role in our subsequent discussion. In order to establish our main result in this section, we require the following hypothesis:

(H) the solution $y(t,t_0,x_0,\omega)$ of (2) exist for all $t \geq t_0$, unique, continuous with respect to the initial data and $||y(t,t_0,X_0,\omega)||$ is locally Lipschitzian in x_0.

THEOREM 1. *Assume that*

(i) *for each* $t \in R^+$, $x \in X$,

$$(4) \quad \lim_{h\to 0^+} R(h,A(t,\omega))x = x,$$

(ii) *the hypothesis* (H) *holds*,

(iii) there exists a measurable map $V(t,x,\omega)$ defined on $R^+ \times X \times \Omega$ into R^+ such that $V(t,x,\omega)$ is locally Lipschitzian in $(t,x) \in R^+ \times X$ ω.p.1,

(iv) *for* $t \in R^+$, $x \in X$ and $\omega \in \Omega$,

$$(5) \quad D^+V(s,y(t,s,x(s,\omega),\omega),\omega)$$

$$= \lim_{h\to 0^+} \sup \frac{1}{h} [V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega)$$

$$+h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,w)$$

$$+ g(s,x(s,\omega),\omega)\},\omega),\omega)$$

$$- V(s,y(t,s,x(s,\omega),\omega),\omega)]$$

$$\leq H_1(s,V(s,y(t,s,x(s,\omega),\omega),\omega),\int_{t_0}^{s} K_0[s,\tau,V(\tau,Y(t,\tau,x(\tau,\omega),\omega)\omega]$$

$$d\tau,\omega) + H_2(s,V(s,y)(t,s,x(s,\omega),\omega),\omega),\omega),$$

where $H_2(t,r,\omega)$, $K_0[t,s,r,\omega]$ *and* $H_1(t,r,\bar{r},\omega)$ are as defined in section 1,

(v) the maximal solution $r(t,t_0,r_0\omega)$ of (3) exists for $t \geq t_0$. Then $x(t,\omega) = x(t,t_0,x_0,\omega)$ is any solution of (1) we have

$$(6) \quad V(t,x(t,\omega),\omega) \leq r(t,t_0,r_0,\omega), \ t \geq t_0,$$

provided

(7) $\quad V(t_0, y(t,t_0,x_0,\omega),\omega) \leq r_0(\omega)$.

Proof. Define

$$m(s,\omega) = V(s, y(t,s,x(s,\omega),\omega),\omega) \text{ for } t_0 \leq s \leq t,$$

so that

$$m(t_0,\omega) = V(t_0, y(t,t_0,x_{0,}\omega),\omega).$$

Since $x(t,\omega)$ is a sample solution of (1) and $V(t,x,\omega)$ satisfies hypothesis (iii), we conclude that $m(s,\omega)$ is sample absolutely continuous W.p.1 for $t \geq t_0$ (see [4] and a particular reference given therein). For small $h > 0$, we have

(8) $\quad m(s+h,\omega)-m(s,\omega)$

$$= V(s+h, y(t,s+h,x(s+h,\omega),\omega),\omega) - V(s, y(t,s,x(s,\omega),\omega),\omega)$$

$$= V(s+h, y(t,s+h,x(s+h,\omega),\omega),\omega) - V(s+h, y(t,s+h,R(h,A(s,\omega))x(s,\omega)$$
$$+h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)$$

$$+ V(s+h, y(t,s+h,R(h,A(s,\omega))x(s,\omega)$$

$$+h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)$$
$$-V(s, y(t,s,x(s,\omega),\omega),\omega)$$

$$\leq L\|y(t,s+h,x(s+h,\omega),\omega) - y(t,s+h,R(h,A(s,\omega))x(s,\omega)$$
$$+h\{f(s,x(s,\omega),\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega)]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega)\|$$

$$+ V(s+h, y(t,s+h,R(h,A(s,\omega))x(s,\omega) + h\{f(s,x(s,\omega),$$

$$\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)$$

$$-V(s,y(t,s,x(s,\omega),\omega),\omega)$$

$$\leq LM||x(s+h,\omega) - R(h,A(s,\omega))x(s,\omega) - h\{f(s,x(s,\omega),$$

$$\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\}||$$

$$+V(s+h,y(t,s+h,R(h,A(s,\omega))x(s,\omega) + h\{f(s,x(s,\omega),$$

$$\int_{t_0}^{s} K[s,\tau,x(\tau,\omega),\omega]d\tau,\omega) + g(s,x(s,\omega),\omega)\},\omega),\omega)$$

$$-V(s,y(t,s,x(s,\omega),\omega),\omega),$$

where L and M are positive constants. Since $x(t,\omega) \in D[A(t,\omega)]$, we have

$$R(h,A(t,\omega))\ [I-hA(t,\omega)]x(t,\omega) = x(t,\omega).$$

Hence

$$(9)\quad R(h,A(t,\omega))x(t,\omega) = x(t,\omega) + hA(t,\omega)x(t,\omega)$$
$$+h[R(h,A(t,\omega))A(t,\omega)x(t,\omega)-A(t,\omega)x(t,\omega)].$$

Using (9) in (8) we have

$$(10)\quad m(s+h),\omega\ )-m(s,\omega\ ) \leq LM||\ x(s+h,\omega)-x(s,\omega)-hA(s,\omega)x(s,\omega)$$
$$-h\ [R(h,A(s,\omega))A(s,\omega)x(s,\omega)-A(s,\omega)x(s,\omega)]\ \ -h\ \{f(s,x(s,\omega),$$
$$\int_{t_0}^{s} K\ [\ s,\tau,x\ (\tau,\omega),\omega,\ \omega)\ +g(s,x(s,\omega),\omega)\}||+V(s+h,y(t,s+h,$$
$$R(h,A(s,\omega))\ x(s,\omega)+h\ \{f(s,x(s,\omega),\ \ \int_{t_0}^{s} K\ [s,\tau,x(\tau,\ \omega),\omega)]d\tau,\omega)$$
$$+g(\ s,x(s,\omega)\omega\},\ \omega),\ \omega),-\ V(s,y(t,s,x(s,\omega),\omega),\omega).$$

Now, using (10),(1) and the assumption given in the statement of the theorem, and sample absolute continuity of $m(s,\omega)$ yeilds the inequality

$$(11)\quad m'(s,\omega) \leq H_1(s, m(s,\omega), \int_{t_0}^{s} K_0[s,\tau, m(\tau,\omega)\ \omega]d\tau, \omega\ )$$

$$+ H_2(s, m(s,\omega),\omega)$$

almost everywhere on $(s,\omega) \in R^+ X \Omega$. From (7) and the definition of $m(s,\omega)$ we have $m(t_0,\omega) \leq r_0\ (\omega)$. Now by following the standard argument analogous to deterministic Theorem 1 in Ref [13, p.192] with suitable modification yields

$$(12)\quad m(s,\omega) \leq r(s,t_0,r_0,\omega),\ t_0 \leq s \leq t$$

Provided $m(t_0,\omega) \leq r_0(\omega)$. Since

$$m(t,\omega) = V(t,y(t,t,x(t,\omega),\omega),\omega) = V(t,x(t,\omega),\omega),$$

the desired result (6) follows from (12) by setting $s = t$. The proof is complete.

It is interesting to make the following observations ·

(a) Taking $r_0(\omega) = V(t,t_0,x_0,\omega),\omega)$, the inequality (6) becames

$$(13)\quad V(t,x(t,\omega),\omega) \geq r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega), \qquad t \geq t_0,$$

which shows the connection between the solutions of systems (1) and (2) in terms of the maxial solutions of (3).

(b) The trivial function

$$A(t,\omega)y(t,\omega) + f(t,y(t,\omega), \int_{t_0}^{t} K\ [t,s;y(s,\omega),\omega\ ]ds,\ \omega) = 0$$

is admissible in Theorem 1 to yield the estimate (6) provded $V(t_0,x_0,\omega) \leq \gamma_0(\omega)$. In this case $y(t,t_0,x_0,\ \omega) = x_0$ and thus the

hypothesis (H) is trivially verified. Since $y(t, t, x, \omega) = x$, the difinition (5) of $D^+V(s,y(t,s,x(s,\omega),\omega),\omega)$ reduces to

$$(14)\quad D^+V(t,x(t,\omega),\omega),\omega) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h,R(h,A(t,\omega)x(t,\omega)$$

$$+h\{f(t,x(t,\omega), \int_{t_0}^{t} K[t,s,x(x,\omega),\omega]ds, \omega)$$

$$+ g(t,x(t,\omega),\omega)\},\omega)-V(t,x(t,\omega),\omega)],$$

which is the usual definition of the generalized derivative of the Lyapunov function relative to the system (1).

We note that Theorem 1 is a direct eextension to stochastic integrodiffeerential systems of the corresponding comparision theorem for deterministic equations recently established by the present author in [13, Theorem 3] which in turn is a further extension of the comparison theorem established by Ladde, Lakshmikantham and Leela in [6]

4.STABILITY RESULTS

In this section employing the comparison theorem developed in the preceding section, we shall present various results giving sufficient conditions for stability in probability and in the mean of the trivial solution of the stockastic integro-differential system (1). Before giving the main results in this section, relative to the stochastic integrodifferential system (2) and the auxiliary stochastic integrodifferential equation (3) we need the corresponding definitions (CSP_1^*) (CSP_2^*),(CSM_1^*)and (CSM_2^*) in our discussion.

For example the connective stability in probability (CSP^*) runs as follows (see, [6], for a similar definition in deterministic case).

Definition 6: The trivial solutions of (2) and (3) are said to be (CSP_1^*) connectively stable in probability, if for each $\varepsilon > 0$, $\eta > 0$, $t_0 \in R^+$, there exists a $\delta = \delta(t_0;\varepsilon\ \eta) > 0$ such that

$$P\ \{\omega:\ ||x_0\ (\omega)\ ||\ > \delta\} < \eta$$

implies

$$P\ \{\omega:\ r(t,\ t_0,\ v(t_0, y(t,t_0,x_0,\omega),\omega),\omega \geq \varepsilon\} < \eta, t \geq t_0.$$

Other definitions may similarly be formulated.

THEOREM 2. *Assume that the hypotehses of Theoreem 1 hold and*

(15) $b(||x||) \leq V(t,x(t,\omega),\omega) \leq a(t,\ ||x||)$,

where $b,\ a(t,.) \in \varkappa$. *Then*

(i) (CSP_1^*) *implies* (SP_2),

(ii) (CSP_2^*) *implies* (SP_1)

Proof. First we prove the statement (i). Let $\eta > 0$, $\varepsilon > 0$, and $t_0 \in R^+$ be given. Assume that (CSP_1^*) holds. Then, given $b(\varepsilon) > 0$, $\eta > 0$, and $t_0 \in R^+$, there exists a positive function $\delta_1 = \delta_1(t_0,\varepsilon,\eta)$ such that

(16) $P\ \{\omega: r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega) \geq b(\varepsilon)\} < \eta, t \geq t_0$

provided

(17) $P\{\omega\ : ||x_0(\omega)|| > \delta_1\} < \eta.$

Let us choose $r_0(\omega)$ so that $V(t_0,y(t,t_0,x_0,,\omega),\omega) = r_0(\omega)$ and

$$r_0(\omega) = a(t_0,||\ x_0(\omega)||\ )$$

Since $a(t_0,.) \in \kappa$ wee can find a $\delta = \delta(t_0, \epsilon\eta) > 0$ such that $P\{\omega : a(t_0, \| x_0(\omega) \|) > \delta_1\} = P\{\omega : \| x_0(\omega) \| > \delta\}$

Now, we claim that (SP_1) holds. Suppose that this claim is false . Then, there would exist a solution process $x(t,\omega)$ of (1) with $P\{\omega : \| x_0(\omega) \| > \delta\} < \eta$ and a $t_1 > t_0$ such that

(18) $P\{\omega : \| x(t_1, \omega) \| \geq \varepsilon\} = \eta$.

By theorem 1, the inquality

(19) $V(t,x(t,\omega),\omega) \leq r(t,t_0,V(t_0,x_0,\omega),\omega),\omega)$

is valid for all $t \geq t_0, \omega \in \Omega$. From (15) and (19) , we have

(20) $b(\| (t,\omega) \|) \leq V(t,x(t,\omega),\omega)$

$\leq r(t,t_0, V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)$

The relations (16), (18) and (20) lead us to the contradiction

$= P\{\omega : \| x(t_1,\omega) \| \geq \varepsilon\}$

$= P\{\omega : b(\| x(t_1,\omega) \| \geq b(\varepsilon)\}$

$\leq P\{\omega : r(t,t_0,V(t_0, y(t,t_0,x_0,\omega),\omega),\omega) \geq b(\varepsilon)\} < \eta,$

thus proving the statement (i).

Now, we shall give the proof of the statement (ii). Assume that (CSP_2^*) holds. Then given $b(\varepsilon) > 0, \eta > 0$, and $t_0 \in R^+$, there exist positive numbers $\delta^0(t_0) = \delta^0$ and $T = T(t_0,\varepsilon,\eta)$ such that(21) $P\{\omega : r(t,t_0 V(t_0,y(t_0 y(t,t_0,x_0,\omega),\omega),\omega) \geq b(\varepsilon)\} < \eta, t \geq t_0+T$ whenever

$P\{\omega : \| x_0(\omega) \| > \delta^0\} < \eta.$

To prove statement (ii) , it is enough to prove that for any $\varepsilon > 0$, $\eta > 0$ and $t_0 \in R^+$ there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\varepsilon,\eta)$ such that $P\{\omega : \| x_0(\omega) \| > \delta_0\} < \eta$ implies

(22) $P\{\omega : \|x(t,\omega)\| \geq \varepsilon\} < \eta, \quad t \geq t_0 + T.$

We claim that (22) holds, otherwise, there exists a sequence $\{t_n\}, t_n \geq t_0 + T$, $t_n \to \infty$ as $n \to \infty$ such that for some solution process of (1) satisfying $P\{\omega: \|x_0(\omega)\| \delta_0\} < \eta$, it will satisfy the relation

(23) $P\{\omega: \|x(t_n, \omega)\| \geq \varepsilon\} = \eta, \quad t_n \geq t_0 + T.$

This together with (19) and (21) will establish the vaalidity of (22) . This completes the proof of the theorem.

Our next theorem establishes the properties of (1) in the mean.

Theorem 3 : Assume that the hypotheses of Theorem 1 hold and (15) *hold with* $b \in V\varkappa$, $a \in C\varkappa$ then

(i) (CSM_1^*) *implies* (SM_1)

(ii) (CSM_2) *implies* (SM_2).

Proof

Let us first prove the statement (i). Let $\varepsilon > 0, t_0 \in R^+$ be given. Assume that (CSM_1^*) holds. Then for given $b(\varepsilon) > 0$ and $t_0 \in R^+$, there exists a positive function $\delta_1 = \delta_1(t_0,\varepsilon)$ such that $E[\|x_0(\omega)\|] \leq \delta_1$ implies

(24) $E[r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)] < b(\varepsilon), \quad t \geq t_0.$

we choose $r_0(\omega)$ such that $V(t_0,y(t,t_0,x_0,\omega),\omega) = r_0(\omega)$ and

$$E[\|x_0(\omega)\|] = a(t_0, E[\|x_0(\omega)\|]).$$

Since $a \in C\varkappa$, we can find a $\delta = \delta(t_0,\epsilon) > 0$ such that

$$E[\|x_0(\omega)\|] \leq \delta \text{ implies } a(t_0, E[\|x_0(\omega)\|]) < \delta_1.$$

Now we claim that if $E\ [||x_0(\omega)||] \leq \delta$, then $E[||x(t,\omega)||] < \varepsilon$, $t \geq t_0$. Suppose that this is false. Then, there would exist a solution process $x(t,\omega)$ of (1) with $E\ [\ ||x_0)\omega)||] \leq \delta$ and a $t_1 > t_0$ such that

(25) $E\ [||\ x(t_1,(\omega)\ ||] = \in$ and $E\ [||\ x(t,\omega)||\ ] \leq \in, t \in [t_0, t_1]$.

By Theorem 1 and (15) we have the inequality (20), and by the convexity of b we have

(26) $b(E[\ ||x(t,\omega)\ ||]) \leq E\ [\ V(t,x(t,\omega),\omega)]$

$\leq E\ [\ r(t,t_0,V(t_0,y(t,t_0,x_0,\omega),\omega),\omega)]$

The relations (24), (25) and (26) lead to the contradiction

$b(\varepsilon) \leq E[V(t_{1,} x(t_1,\omega),\omega)\ ]$

$\leq E[r(t,t_0,V(t_0,y(t_1,t_0,x_0,\omega),\omega),\omega)] < b\ (\in)$,

Providing (i). The proof of (ii) can be formulated similarily .

In concluding this section we note that, one can very easily formulate other definitions and theoremms on stability and boundedness as in [4,5,14] analogously with suitable modifications for the integrodifferental system (1). We omit the details.

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ICHTHYOFAUNA OF THE RIVER GANGA AT THE FOOTHILLS OF GARHWAL HIMALAYA

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ABSTRACT

The paper deals with the Ichthyofauna of the river Ganga at foothills of Garhwal Himalaya. A systematic list of 44 species is given with their present valid names and earlier names given by Day and Hamilton.

Key words and Phrases : Ichthyofauna, Ganga river, Systematic list.

INTRODUCTION

In 1822 Hamilton [5] listed the fish fauna of the river Ganga and its tributaries, no other work has been published before this, Day [4], Hora [7], Hora and Mukurjee [8], Jayaram [9], Mahajan [10], Pant [12] and Venkateshwarlu and Menon [15] have also published reports on the Ichthyofauna of the Ganga river system.

The fish fauna of this river in the uplands have been studied by Heckel [6], Malik [11], Sehgal et al. [13] Badola [1], Badola and Pant [2], Badola and Singh [3] and Singh et al. [14]. But the reports on the fish fauna of the Ganga river at the foothills of Garhwal Himalaya are scanty and limited. The paper deals with the Ichthyofauna of the river Ganga and its tributaries (Song and Suswa) from Lakshman Jhula to Kangri Village, Hardwar (Table).

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TABLE : SYSTEMATIC LIST OF FISHES

Present Scientific Name	Name according to Day	Name according to Hamilton	Local Name
A	B	C	D
	Phylum	: VERTEBRATA	
	Subphylum	: CRANIATA	
	Super class	: GNATHOSTOMATA	
	Series	: PISCES	
	Class	: TELEOSTOMI	
	Sub class	: ACTINOPTERYGII	
	Order	: CYPRINIFORMES	
	Sub order	: CYPRINOIDEI	
	Family	: CYPRINIDAE	
	Sub family	: RASBORINAE	
	Genus	: *Barilius* Hamilton	
Barilius bendelisis (Ham.)	*Barilius bendelisis*	*Cyprinus (Barilius) bendelisis*	Bhola
Barilius bola (Ham.)	*Barilius bola*	*Cyprinus (Barilius) Bola*	Nayar Bhola
Barilius vagra (Ham.)	*Barilius vagra* *Barilius modestus*	*Cyprinus (Barilius) vagra*	persee
	Genus : *Danio*	Hamilton	
Danio (Danio) devario (Ham.)	*Danio devario*	*Cyprinus (Babdio) devario*	Patukari
Danio (Brachydanio) rerio	*Danio rerio*	*Cyprinus (Danio) rerio* *Cyprinus (Danio) chapalio*	
	Genus : *Esomus*	: Swainson	
Esomus danricus (Ham.)	*Nuria danrica*	*Cyprinus (Danio) danrica* *Cyprinus (Danio) sunita* *Cyprinus (danio) jogia*	Denc

Contd..

A	B	C	D
	Genus : *Rasbora* Bleeker		
Rasbora daniconius (Ham.)	*Rasbora daniconius*	*Cyprinus (Danio) daniconius* *Cyprinus (Danio) anjana*	Dendua
	Sub family : Cyprininae		
	Ganus : *Chagunio* Smith		
Chagunio chagunio (Ham.)	*Barbus chagunio*	*Cyprinus (Cyprinus) chagunio*	Gelhari
	Genus : *Crossocheilus* Van hasselt		
Crossocheilus latius latius (Ham.)	*Cirrhina latia*	*Cyprinus (Garra) latius* *Cyprinus (Garra) gohama* *Cyprinus (Garra) sada*	Pet phorni
	Genus : *Garra* Hamilton		
Garra gotyla gotyla (Gray)	*Discognathus lamta* impart	*Cyprinus lamta*	Siltoka
	Genus : *Labeo* Cuvier		
Labeo boga (Ham.)	*Labeo boga*	*Cyprinus (Bangana) boga*	Bhagan
Labeo dero (Ham.)	*Labeo microphthalmus* *Labeo diplostomus* *Labeo sindensis*	*Cyprinus (Bangana) dero*	Arangi
Labeo gonius (Ham.)	*Labeo gonius*	*Cyprinus (Cyprinus) gonius* *Cyprinus (Cyprinus) curchiu* *Cyprinus (Cyprinus) cursa*	kursi Bata

A	B	C	D
	Genus : *Puntius* Hamilton		
Puntius sarana sarana (Ham.)	*Barbus sarana* *Barbus chrysopoma* *Barbus pinnauratus*	*Cyprinus* (*Cyprinus*) *Saran*	Darahee
Puntius sophore (Ham.)	*Barbus sophore* *Barbus stigma* *Barbus puntio* *Barbus chrysopterus*	*Cyprinus* (*puntius*) *sophore* *Cyprinus* (*puntius*) *puntio*	Sidhari
Puntius ticto (Ham.)	*Barbus ticto*	*Cyprinus* (*puntius*) *ticto*	Sidhari
	Barbus punctatus		
	Genus : *Tor* Gray		
Tor putitora	*Barbus tor* (in part	*Cyprinus* (*Cyprinus*) *putitora*	Mahaseer
Tor tor (Ham.)	*Barbus tor* (in part) *Barbus hexasticus*	*Cyprinus* (*Cyprinus*) *tor*	Mahaseer
	Sub family : **Schizothoracinae** Heckel		
	Genus : *Schizothorax* Heckel		
Schizotnorax plagiostomus	*Oreinus plagiostomus*		Dhibrua
Schizothorax sinuatus	*Oreinus sinuatus*		Maseen

Contd..

A	B	C	D
	Genus : *Schizothoraichthys* Misra		
Schizothoraichthys progastus (mc Clelland)	*Schizothorax progastus*		Chonga
	Family : **Cobitidae**		
	Genus : *Botia* Gray		
Botia dario (Ham.)	*Botia dario*	*Cabitus geto*	Baghaua
	Genus : *Nemacheilus* Van Hasselt		
Nemacheilus beavani (Gunther)	*Nemacheilus beavani*		Natwa
Nemacheilus botia (Ham.)	*Nemacheilus botia*	*Cobitus botia* *Cobitus bilturio*	Natwa
Nemacheilus montanus (Mc Clelland)	*Namacheilus denisoni*		Natwa
Nemacheilus rupicola (Mc Clelland)	*Nemacheilus rupicola*		Natwa
Nemacheilus savona	*Nemacheilus savona*	*Cobitus turio*	Natwa
	Division : SILURI SUB order ; SILUROIDEI Family : Bagridae		
	Genus : *Mystus* Gronow (emend. Scopoli)		
Mystus cavasius (Ham.)	*Macrones cavasius*	*Pimelodus cavasius*	Sutahawa Tengara
Mystus tengara (Ham.)	*Macrones tengara*	*Pimelodus tengara*	Tengana
Mystus seenghala (Sykes)	*Macrones seenghala*		Seenghara

Contd..

A	B	C	D
	Genus : *Rita* Bleeker		
Rita rita (Ham.)	*Rita buchanani*	*Pimelodus rita*	Hunna Rita
	Family : **Sisoridae**		
	Genus : *Bagarius* Bleeker		
Bagarius bagarius	*Bagarius yarrelli*	*Pimelodus bagarius*	Gonch
	Genus : *Glyptothorax* Blyth		
Glytothorax conirostris (*Steind*)	*Glyptosternum conirostri* *Glyptosternum lonah* (in part)		Nau
Glyptothorax pectinopterusrus (Mc Clelland)	*Glyptosternum pectinopterum*		Nau
Glyptothorax madraspatanum	*Glyptosternum modestum*		Nau
	Family : **Schilbeidae**		
	Genus : *Clupiscma* Swainson		
Clupisoma garua (Ham.)	*Pseudeutropius garua*	*Silurus* (*Callichrous*) *garua*	Baikari Karahi
	Order : BELONIFORMES		
	Suborder: SCOMERBOSOCOIDEI		
	Family : **Belonidae**		
	Genus : *Xenentodon* Regan		
Xenentodon cancila (Ham.)	*Belone cancila*	*Esor cancila*	Kauva

Contd..

A	B	C	D
	Order : OPHIOCEPHALIFORMES (CHANNIFORMES) Family : **Ophiocephalidae (Channidae)** Genus : *Channa* Scopoli		
Channa marulius (Ham.)	*Ophiocephalus marulius*	*Ophiocephalus marulius*	Saur
Channa punctatus (Bloch)	*Ophiocephalus punctatus*	*Ophiocephalus lata*	Girai
Channa striatua (Bloch)	*Ophiocephalus striatus*	*Ophocephalus wrahl* *Ophiocephalus chena*	Saur
Channa gachua (Ham.)	*Ophiocephalusgachua*	*Ophiocephalus gachua*	Chanaga
	Order : PERCIFORMES Suborder: ANABANTOIDEI Family : **Anabantidae** Ganus : *Colisa* Cuvier et valenciennes		
Colisa fasciatus (Block et schn.)	*Trichogaster fasciatus*	*Trichopodus colisa* *Trichopodius bejeus* *Trichopodius cotra*	Khosti
	Order : MASTACEMBELIFORMES Family : **Mastacembelidae** Genus : *Mastacembelus* Schopoli		
Mastacembelus armatus (Lacepede)	*Mastacembelus armatus*	*Macrognathus armatus*	Baam
Mastacembelus pancalus (Ham.)	*Mastacembelus pancalus*	*Macrognathus pancalus*	Malga Pataya

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COINCIDENCE THEOREMS IN LINEAR TOPOLOGICAL SPACES

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ABSTRACT

The notion of diminishing joint orbital diametral sum for a pair of commuting mapping has been generalized to a Hausdorff locally convex space whose topology is generated by a family of seminorms. Subsequently, this notion is utilized to prove some coincidence theorems which generalize several known results.

AMS(MOS) Subject classifications(1980): 47H10, 54H25.

Key words and phrases : Coincidence theorems, locally convex linear topological spaces, diminishing joint orbital diametral sum.

1. INTRODUCTION

Several fixed point theorems for nonexpansive mappings in locally convex linear topological spaces are known. In this setting the notions of diminishing orbital diameters and normal structure were given by Tan [7]. These notions were the generalizations of the corresponding notions given earlier by Belluce and Kirk [1] and Brodskii and Milman [2]. Recently the notions of diminishing joint orbital diameteral sum and joint normal structure in a metric space were introduced by Mishra [5] and several coincidence theorems for a pair of commuting mappings on metric and Banach spaces were proved. In this paper it is shown that

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most of these results hold for the same class of mappings on a bounded subset of a Hausdorff locally convex linear topological space.

Let (E,τ) be a locally convex linear topological space. A family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms on E is said to be an associated family of seminorms for τ if the family $B = \{tU : t > 0\}$ where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x \in E : p_{\alpha_i}(x) < 1\}$ forms a base of neighbourhoods of zero for τ. The set U is also given by $U = \{x \in E : p(x) < 1\}$ where p is the seminorm max $\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\}$. A family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms on E is called an augmented associated family of seminorms for τ if it is an associated family for τ and has further property that given $\beta, \gamma \in \Lambda$ the seminorm max $\{p_\beta, p_\gamma\} \in \{p_\alpha : \alpha \in \Lambda\}$. We shall denote an associated family and an augmented associated family for τ by D and D* respectively. It is well-known that given a locally convex linear topological space (E,τ), there always exists a family of seminorms $\{p_\alpha : \alpha \in \Lambda\}$ such that $\{p_\alpha : \alpha \; \Lambda\} = D^*$. For details we refer to Köthe [4]. Unless otherwise stated, throughout this paper, E will denote a Hausdorff locally convex linear topological space (E,τ) defined by $\{p_\alpha : \alpha \in \; \} = D^*$.

DEFINITION 1.1 [9]. Let M be a nonempty subset of E. A mapping $f : M \to M$ is called D-nonexpansive (resp. D*-nonexpansive) if for all $x, y \in M$ and for each $p_\alpha \in D$ (resp. for each $p_\alpha \in D^*$), $p_\alpha(f(x) - f(y)) \leq p_\alpha(x-y)$.

Since f is D-nonexpansive $\Longleftrightarrow$ f is D*-nonexpansive (cf. Tarafdar [8]), we shall simply say f is nonexpansive. For equivalent difinitions of nonexpansive mappings, we refer to Srivastava and Srivastava [6], Tan [7] and Taylor [10].

Let S and T be commuting mappings of E into E, and let $f = ST = TS$. For each $x \in E$, let $0(f^n(x)) = \{f^n(x), f^{n+1}(x), \ldots\}$ where $n = 0, 1, 2, \ldots$ and $f^0(x) = x$. For $x, y, \in E$ and $\alpha \in \Lambda$, define

$$P_\alpha(x,y) = p_\alpha(Tx-Sx) + p_\alpha(Ty-Sy) + p_\alpha(Tx-Sy) + p_\alpha(Ty-Sx) + p_\alpha(Tx-Ty).$$

It is clear that $P_\alpha(x,y) = P_\alpha(y,x)$. For each nonempty subset A of E, define $J_\alpha(A) = \sup\{P_\alpha(x,y): x,y \in A\}$. Further, suppose that $R_\alpha(f^n(x)) = J_\alpha(0(f^n(x)))$ and $r_\alpha(x) = \lim_n R(f^n(x))$. Then we have the following:

DEFINITION 1.2. If for each $x \in E$, there is an $\alpha \in \Lambda$ such that $J_\alpha(0(x)) < \infty$ and $r_\alpha(x) < J_\alpha(0(x))$ whenever $J_\alpha(0(x)) > 0$, then mappings $S, T : E \to E$ will be said to have diminishing joint orbital diametral sum (dimjodis) on E.

DEFINITION 1.3.(i) Let A be a bounded subset of E. A point $a \in A$ will be called a joint nondiametral point of A (with respect to $S,T : E \to E$) if there is an $\alpha \in \Lambda$ such that $\sup\{P_\alpha(a,x) : x \in A\} < J_\alpha(A)$.

(ii) A bounded subset K of E will be said to have joint normal structure (with respect to $S, T : E \to E$) if for each convex subset H of K which contains more than one point, there is a joint nondiametrial point of H.

For equivalent definitions in metric spaces, we refer to [5].

Let K be a nonempty subset of E. A mapping $g:K \to K$ is called affine if $g(kx+(1-k)y) = k\,gx + (1-k)y$ for all $x,y \in K$ and for all $k \in [0,1]$.

PROPOSITION 1.4. Let $S, T : E \to E$, and let $A, B \subseteq E$ such that $A \subseteq B$. Then for each $\alpha \in \Lambda$,

(i) $J_\alpha(A) \leq J_\alpha(B)$,

(ii) $J_\alpha(A) = J_\alpha(\bar{A})$, if S and T are continuous, where $\bar{A}$ is the colsure of A.

PROOF. The proof of (i) is obvious. We shall prove (ii). Let $\alpha \in \Lambda$ be arbitrary. Since $A \subseteq \bar{A}$ it follows that $J_\alpha(A) \leq J_\alpha(\bar{A})$. To prove that $J_\alpha(\bar{A}) \leq J_\alpha(A)$, let $x, y \in \bar{A}$, and let $\epsilon > 0$. Since S and T are continuous, we may choose u,v in A such that $p_\alpha(Tx-Tu) < \epsilon$, $p_\alpha(Ty-Tv) < \epsilon$, $p_\alpha(Sx-Su) < \epsilon$, and $p_\alpha(Sy-Sv) < \epsilon$. Hence using the definition of $p_\alpha(x,y)$ and triangle inequality, it can be easily verified that

$$\begin{aligned} P_\alpha(x,y) &\leq P_\alpha(u,v) + 3\,(p_\alpha(Tx-Tu) + p_\alpha(Ty-Tv)) + 2(p_\alpha(Sx-Su) \\ &\qquad + p_\alpha(Sy-Sv)) \\ &\leq P_\alpha(u,v) + 10\,\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $P_\alpha(x,y) \leq P_\alpha(u,v)$. Hence $J_\alpha(\bar{A}) \leq J_\alpha(A)$, proving (ii).

PROPOSITION 1.5: Let A be a subset of E, and let $S, T: E \to E$. Then for each $\alpha \in \Lambda$,

(i) $J_\alpha(A) = J_\alpha(A)_c$, if S and T are affine,

(ii) $J_\alpha(A) = J_\alpha\overline{(A)}_c$, if S and T are affine and continuous; where $(A)_c$ and $\overline{(A)}_c$ are the convex hull and closed convex hull of A respectively.

PROOF. We shall prove (i) as (ii) follows from (i) and 1.4 (ii). Since $A \subseteq (A)_c$ we have from 1.4 (i) that $J_\alpha(A) \leq J_\alpha(A)_c$. To prove that $J_\alpha(A)_c \leq J_\alpha(A)$, let x,y $\in (A)_c$. Then we may assume that $\Lambda = \Sigma\, a_i x_i$ and $y = \Sigma\, b_j x_j$

and $y = b_j x_j$ where $a_i, b_j > 0$, $\sum a_i = 1 = \sum b_j$ and $x_i, x_j \in A$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Then following [5] it can be shown that

$$p_\alpha (Tx-Sx) \leq \Sigma\, b_j\, \Sigma a_i\, p_\alpha (Tx_i - Sx_i) ,$$
$$p_\alpha (Ty-Sy) \leq \Sigma\, b_j\, \Sigma a_i\, p_\alpha (Tx_j - Sx_j) ,$$
$$p_\alpha (Tx-Sy) \leq \Sigma\, b_j\, \Sigma a_i\, p_\alpha (Tx_i - Sx_j) ,$$
$$p_\alpha (Ty-Sx) \leq \Sigma\, b_j\, \Sigma\, a_i\, p_\alpha (Tx_j - Sx_i),$$
$$p_\alpha (Tx-Ty) \leq \Sigma\, b_j\, \Sigma\, a_i\, p_\alpha (Tx_i - Tx_j)$$

for each $\alpha \in \Lambda$.

Therefore $P_\alpha(x,y) \leq (\Sigma\, b_j)\ (\Sigma\ a_i\ P_\alpha(x_i,x_j)) \leq (\Sigma b_j)$ $(\Sigma\ a_i)\ J_\alpha\ (A)$. Hence $J_\alpha\ (A)_c \leq J_\alpha (A)$. This proves that $J_\alpha\ (A) = J_\alpha\ (A)_c$.

2. MAPPINGS WITH DIMJODIS

In this section we shall prove two coincidence theorems for mappings with diminishing joint orbital diametral sum (dimjodis) and with normal structure. The results obtained herein generalize certain theorems of Mishra [5, Theorems 3.2 and 3.3] and Tan [7].

DEFINITION 2.1. Let $a \in E$. Define $B_\alpha\ (a,r) = \{x \in E\ :\ P_\alpha(a,x) \leq r\}$, $r > 0$ and $\alpha \in \Lambda$.

THEOREM 2.2. Let $K \subseteq E$ be nonempty bounded closed convex, and let $M \subseteq K$ be nonempty weakly compact. If S and T are non-expansive commuting affine mappings of K into K such that

(i) $\overline{O(x)_c} \cap M \neq \phi$ for each $x \in K$, and

(ii) S and T have *dimjodis* on K,

then there exists a $z \in M$ such that $Sz = Tz$ i.e. S and T have a coincidence point z in M.

PROOF. Let $f = ST = TS$. A nonempty subset K^* of K minimal with respect to being closed convex f-invariant and having points in common with M is obtained by using weak compactness of M and the Zorn's lemma. Let $M^*=K^* \cap M$.

Let $x \in K^*$ and $\alpha \in \Lambda$. Then we wish to show that $J_\alpha(0(x)) = 0$. Suppose $J_\alpha(0(x)) > 0$. By (ii), there exists an integer N such that $k_\alpha = J_\alpha(0(f^n(x)))<J_\alpha(0(x))$ Suppose $L = \{z \in K^* : p_\alpha(z,f^N(x)) \leq k_\alpha$ for almost all $n\}$. Then $L \subseteq K^*$. We shall show that L is nonempty convex invariant under f and has points in common with M. Taking $f^m(x) \in 0(f^N(x))$, i.e. $m \geq N$ we have $p_\alpha(f^m(x),f^n(x)) \leq k_\alpha$ for all n, $m \geq N$. Hence $f^m(x) \in L$ and thus $L \neq \phi$. Suppose $y \in L$. Then for some intger N_1 we have $p_\alpha(y,f^n(x)) \leq k_\alpha$ for all $n \geq N_1$. Consider

$$\begin{aligned} p_\alpha(f(y),f^{n+1}(x)) = p_\alpha & (Tf(y) - Sf(y) + p_\alpha(Tf^{n+1}(x)-Sf^{n+1}(x)) \\ & + p_\alpha(Tf(y) - Sf^{n+1}(x)) + p_\alpha(Tf^{n+1}(x)-Sf(y)) \\ & + p_\alpha(Tf(y) - Tf^{n+1}(x), \end{aligned}$$

Since f is nonexpansive (as S and T are so) and S and T are commuting, it follows that

$$\begin{aligned} p_\alpha(f(y),f^{n+1}(x)) & \leq p_\alpha(Ty-Sy) + p_\alpha(Tf^n(x) - Sf^n(x)) \\ & + p_\alpha(Ty-Sf^n(x) + p_\alpha(Tf^n(x) - Sy) \\ & + p_\alpha(Ty - Tf^n(x)) \\ & = p_\alpha(y,f^n(x) \leq k_\alpha \end{aligned}$$

for all $n + 1 \geq N_1 + 1$. Hence $f(y) \in L$ and thus f maps L inot L. It follows from the continuity of f that $f(\bar{L}) \subseteq \overline{f(L)} \subseteq \bar{L}$. Therefore f maps L into L. Also $0(f^N(x)) \subseteq \bar{L}$, if follows that $\overline{0(f^N(x))}_c \subseteq \bar{L}$. Therefore by (ii) we have $\bar{L} \cap M \neq \phi$. Hence the minimality of K* implies that $K^* = \bar{L}$.

Now let $u \in K^*$. Then since $u \in \bar{L}$, for $\epsilon > 0$ there exists a point u^* inL such that $p_\alpha(u-u^*) < \epsilon/5$. Moreover, we can find an N_2 such that $p_\alpha(u^*, f^n(x)) \leq k_\alpha$ for all $n \geq N_2$. Further,

$$
\begin{aligned}
P_\alpha\,(u, f^n(x)) = {} & p_\alpha\,(Tu-Su) + p_\alpha(Tf^n(x)-Sf^n(x)) + p_\alpha\,(Tu-Sf^n(x)) \\
& + p_\alpha(Tf^n(x)-Su) + p_\alpha(Tu-Tf^n(x)) \\
\leq {} & p_\alpha(Tu-Tu^*) + p_\alpha(Tu^*-Su^*) + p_\alpha(Su^*-Su) \\
& + p_\alpha(Tf^n(x)-Sf^n(x)) + p_\alpha(Tu-Tu^*) \\
& + p_\alpha(Tu^*-Sf^n(x)) + p_\alpha\,(Tf^n(x) - Su^*) \\
& + p_\alpha(Su^*-Su) + p_\alpha(Tu-Tu^*) + p_\alpha(Tu^*-Tf^n(x)).
\end{aligned}
$$

No using the nonexpansiveness of S and T it can be easily verified that $P_\alpha\,(u, f^n(x)) \leq P_\alpha(u^*, f^n)) + 5\,p_\alpha(u-u^*) \leq k_\alpha + \epsilon$ for all $n \geq N_2$.

Let $B_\alpha\,(u, k_\alpha + \epsilon) = \{y \in E : p_\alpha(y,u) \leq k_\alpha + \epsilon\}$. Then we observe that $0(f^n(x)) \subseteq B_\alpha\,(u, k_\alpha + \epsilon)$ for all $n \geq N_2$. Since S and T are continuous and affine, $B_\alpha(u, t_\alpha + \epsilon)$ is closed and convex and hence $\overline{0(f^n(x))}_c \subseteq B_\alpha(u, k_\alpha + \epsilon)$ for all $n \geq N_2$. Also by (i) $\overline{0(f^n(x))}_c \cap M^* \neq \phi$ for al $n \geq N_2$. Hence the weak compactness of M* ensures the existence of a point t in $\bigcap_{n=N_2}^{\infty} \overline{0(f^n(x))}_c \cap M^*$ and consequently $t \in B_\alpha(u, k_\alpha + \epsilon)$. Since $\epsilon > 0$ and $u\ (\in K^*)$ are arbitrary, $t \in B_\alpha(u, k_\alpha)$

for all $u \in K^*$. Hence the set $A = \{z \in K^* : K^* \subseteq B_\alpha(u,k_\alpha)\}$ is nonempty. Since S and T are affine and continuous, A, $\subseteq K^*$ is closed and onvex.

We shall show that $f(A) \subseteq A$. Suppose this is not the case. Then $f(z) \notin A$ for some $z \in A$. Clearly $H = B_\alpha(f(z), k_\alpha) \cap K^*$ is closed and convex. Since $z \notin A$, we have $z \in K^* \subseteq B_\alpha(Z,k_\alpha)$. Also $f(z) \in K^*$. Hence $f(z) \in B_\alpha(z,k_\alpha)$. Therefore $z \in H$ and H is nonempty. Since $f(z) \notin A$, there exists $a \in K^*$ such that $y \notin B_\alpha(f(z),k_\alpha)$. This implies that $P_\alpha(f(z),y) > k_\alpha$. Hence H is a propφer subset of K^*. Let $y^* \in K^*$. Then $f(y^*) \in K^*$. Also $y^* \in K^* \subseteq B_\alpha(z,k_\alpha)$, it follows that $P_\alpha(y^*,z) \leq k_\alpha$. It can be easily verified that $P_\alpha(f(z),f(y^*)) \leq P_\alpha(z,y^*) \leq k_\alpha$. Therefore $f(y^*) \in H$, proving $f(H) \subseteq H$. Moreover, since $z \in H$, $\overline{O(z)}_c \subseteq H$ and hence from (i) $H \cap M \neq \phi$. This shows that H is a nonempty closed convex proper subset of K^* which is invariant under f and has points in common with M, a contradiction to the minimality of K^*. Hence $f(A) \subseteq A$.

Let $v,w \in A$. Then $P_\alpha(v,w) \leq k_\alpha$. Consequntly, $J_\alpha(A) \leq k_\alpha = J_\alpha\ (O(f^N(x))) \leq J_\alpha(O(x)) < J_\alpha(K^*)$. Therefore A is a proper subset of K^*. Hence our assumption that $J_\alpha(O(x)) > 0$ is incorrect. Therefore $J_\alpha(O(x)) = 0$ and $Sx = Tx$.

THEOREM 2.3. Let $K \subseteq E$ be nonempty bounded closed convex, and let $M \subseteq K$ be nonempty weakly compact. If S and T are non-expansive commuting affine mappings of K into K such that for each $x \in K$,

(i) $\overline{(O(x))}_c \cap M \neq \phi$, and

(ii) $\overline{(O(x))}_c$ has joint normal structure,

then there exists a point $z \in M$ such that $f(x)=z$, where $f= ST = TS$.

PROOF. Define K^* as in the proof of Theorem 2.2. We shall show that for each $x \in K^*$, $0(x)$ is a singleton. Suppose that there is an $x \in K^*$ such that $0(x)$ contains more than one point. Then by (ii), there exists a point $y \in \overline{(0(x))}_c$ such that y is a joint nondiametral point (with respect to S and T). That is there exists an $x \in \Lambda$ such that $\sup \{P_\alpha(y,z) : z \in \overline{(0(z))}_c\} = s_\alpha < J_\alpha\overline{(0(x))}_c) = J_\alpha(0(x))\}$. Let $A = \{z \in K^* : K^* \subseteq B_\alpha(z, s_\alpha)\}$. It can be shown as in the proof of Theorem 2.2 that A is closed convex, $A \cap M \neq \phi$ and $f(A) \subseteq A$. By the minimality of K^* it follows that $A = K^*$. Hence $J_\alpha(K^*) = J_\alpha(A) \leq s_\alpha < J_\alpha(0(x)) \leq J_\alpha(K^*)$ which is absured. Hence for each $x \in K^*$, $0(x)$ is a singleton. Hence $f(x) = x$ for all $x \in K^*$.

3. BOUNDED MAPPINGS

In this section we shall discuss further properties of mappings with diminishing joint orbital diametral sum (dimjodis) by using the notion of bounded mappings. Such mappings were first studied by Kirk [3]. The term bounded mappings is due to Tan [7].

DEFINITION 3.1. Let $K \subseteq E$ be nonempty. A mapping $f : K \to K$ is said to be bounded if and only if for each $\alpha \in \Lambda$ there is a real number $c_\alpha > 0$ such that $p_\alpha(f^k(x), f^k(y)) \leq c_\alpha p_\alpha(x-y)$ for all $x, y \in K$ and all $k = 1, 2, \ldots$.

In the above condition if we set $f = ST = TS$, where S and T are self mappings of K and $c_\alpha = 1$, then f is nonexpansive and this happens when both S and T are nonexpansive.

PROPOSITION 3.2. Let $K \subseteq E$ be nonempty bounded, and let S and T be commuting continuous mappings of K into K such that $f = ST = TS$ is bounded. Then for each $\alpha \in \Lambda$ the function $x \to R_\alpha(x) = J_\alpha(0(x))$ is continuous.

PROOF. Let $\{x_\lambda : \lambda \in I\}$ be a net in K such that $x_\lambda \to x$ for some $x \in K$. Let I* be a directed subset (directed by $\leq$) of I. To prove that the given function is continuous we shall whow that $R_\alpha(x_\lambda) \to R_\alpha(x)$ for each $\alpha \in \Lambda$. Consider

$$|R_\alpha(x_\lambda) - R_\alpha(x)| = |\sup\{ P_\alpha(xy) : x,y \in O(x_\lambda)$$
$$-\sup\{P_\alpha(x,y) : x,y \in O(x)\ |$$

Further,

$$\sup\{P_\alpha(x,y) : x,y, \in O(x_\lambda)\}$$
$$= \sup_{i,j \geq 0} \{P_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda))$$
$$+ p_\alpha(Tf^j(x_\lambda) - Sf^j(x_\lambda)) + p_\alpha(Tf^i(x_\lambda) - Sf^j(x_\lambda))$$
$$+ p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) + p_\alpha(Tf^i(x_\lambda) - Tf^j(x_\lambda))\},$$

$$\sup\{P_\alpha(x,y) : x,y \in O(x)\}$$
$$= \sup_{i,j \geq 0} \{p_\alpha(Tf^i(x) - Sf^i(x)) + p_\alpha(Tf^j(x) - Sf^j(x))$$
$$+ p_\alpha(Tf^i(x) - Sf^j(x)) + p_\alpha(Tf^j(x) - Sf^i(x))$$
$$+ p_\alpha(Tf^i(x) - Tf^j(x))\}.$$

Therefore,

$$|R_\alpha(x_\lambda) - R_\alpha(x)| \leq \sup_{i,j \geq 0} \{A_1 + A_2 + A_3 + A_4 + A_5\}$$

where

$$A_1 = |p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^i(x) - Sf^i(x))|,$$
$$A_2 = |p_\alpha(Tf^j(x_\lambda) - Sf^j(x_\lambda)) - p_\alpha(Tf^j(x)) - Sf^j(x))|,$$
$$A_3 = |p_\alpha(Tf^i(x_\lambda) - Sf^j(x_\lambda)) - p_\alpha(Tf^i(x)) - Sf^j(x))|,$$
$$A_4 = |p_\alpha(Tf^j(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^j(x) - Sf^i(x))|,$$
$$A_5 = |p_\alpha(Tf^i(x_\lambda) - Tf^j(x_\lambda)) - p_\alpha(Tf^i(x) - Tf^j(x))|.$$

Let $\epsilon > 0$ and set $\xi = \min \{\epsilon/_{10}, \epsilon/_{10c_\alpha}\}$. Since $x_\lambda \to x$ and S and T are continuous, there exists $\lambda_o \in I^*$ such that for all $\lambda > \lambda_o$ we have

$$\text{(3.2.1)} \quad \begin{aligned} &p_\alpha(x_\lambda - x) < \xi, \\ &p_\alpha(Tx_\lambda - Tx) < \xi, \\ &p_\alpha(Tx_\lambda - Sx) < \xi. \end{aligned}$$

Further,

$$\begin{aligned} p_\alpha & (Tf^i(x_\lambda) - Sf^i(x_\lambda)) \\ &\leq p_\alpha (Tf^i(x_\lambda) - Tf^i(x)) + p_\alpha(Tf^i(x) - Sf^i(x)) \\ &\quad + p_\alpha(Sf^i(x) - Sf^i(x_\lambda)), \end{aligned}$$

and

$$\begin{aligned} p_\alpha & (Tf^i(x_\lambda) - Tf^i(x)) \\ &= p_\alpha(f^i(Tx_\lambda) - f^i(Tx)) \leq c_\alpha p_\alpha(Tx - Tx), \\ & p_\alpha (Sf^i(x_\lambda) - Sf^i(x)) \leq c_\alpha p_\alpha(Sx - Sx). \end{aligned}$$

Therefore,

$$\text{(3.2.2)} \quad \begin{aligned} &|p_\alpha(Tf^i(x_\lambda) - Sf^i(x_\lambda)) - p_\alpha(Tf^i(x) - Sf^i(x))| \\ &\leq c_\alpha \{p_\alpha(Tx_\lambda - Tx) + p_\alpha(Sx_\lambda - Sx)\}, \end{aligned}$$

..... Similiarly,

$$\text{(3.2.3)} \quad \begin{aligned} &|p_\alpha(Tx_\lambda - Sx_\lambda) - p_\alpha(Tx_\lambda - Sx)| \\ &\leq p_\alpha((Tx_\lambda - Tx) + p_\alpha(Sx_\lambda - Sx). \end{aligned}$$

Therefore from (3.2.1) through (3.2.3) it follows that $A_1 < \epsilon/5$ for all $i \geq 0$ and for all $\lambda > \lambda_o$. Similarly, $A_2 < \epsilon/5$ for all $i \geq 0$ and for all $\lambda > \lambda_o$. Using similar arguments, each of A_3, A_4 and A_5 can be made less than $\epsilon/5$ for all i, j, ≥ 0 and for all $\lambda > \lambda_o$. Consequently,

$$|R_\alpha(x_\lambda) - R_\alpha(x)| < \epsilon \text{ for all } \lambda \geq \lambda_0.$$

Therefore $R_\alpha(x_\lambda) \to R_\alpha(x)$.

COROLLARY 3.3. Let $K \subseteq E$ be nonempty bounded, and S and T be commuting nonexpansive mappings of K into K. Then the function $x \to R_\alpha(x) = J_\alpha(0(x))$ is continuous for each $\alpha \in \Lambda$.

THEOREM 3.4. Let $K \subseteq E$ be nonempty bounded, let S and T be commuting continuous mappings of K into K such that (3.4.1) S and T have *dimjodis*.

(3.4.2) f (= ST=TS) is bounded,

(3.4.3) there is an $x \in K$ such that a subsequency of the sequence $\{f^n(x)\}$ converges to a point $z \in K$.

Then $\lim_n f^n(x) = z$ and z is a coincidence point of S ant T, i.e. $Sz = Tz$.

PROOF. Let $\{f^{n(s)}(x)\}$ be a subequence of $\{f^n(x)\}$ with its limit z ($z \in K$). Then since $R_\alpha(x)$ is continuous, for each $\alpha \in \Lambda$, it follows that $\lim_n R_\alpha(f^{n(s)}(x) = R_\alpha(z)$. Also $\lim_n R_\alpha(f^n(x)) = r_\alpha(x)$ and $\{R_\alpha(f^{n(s)}(x)\}$ is a subsequence of $\{R_\alpha(f^n(x)\}$, if follows that

(3.4.4) $\lim R_\alpha(f^{n(s)}(x)) = r_\alpha(x) = R_\alpha(z)$.

Since $\lim_n f^{n(s)}(x) = z$ and S and T are continuous, for $\epsilon > 0$ there exists a positive integer m such that for all $s \geq m$,

$$p_\alpha(f^{n(s)}(x) - z) < \epsilon/10\, c_\alpha,$$

$$p_\alpha(f^{n(s)}(Tx) - Tx) < \epsilon/10\, c_\alpha,$$

$$p_\alpha(f^{n(s)}(S\,;\, - Sx) < \epsilon/10\, c_\alpha,$$

Since f is bounded, we have for all $t \geq 1$,

$$(3.4.5) \quad p_\alpha(f^{n(s)+t}(Tx) - f^t(Tz)) < \epsilon/10$$

$$p_\alpha(f^{n(s)+t}(Sx) - f^t(Sz) < \epsilon/10.$$

Let N be a positive integer. Then

$$(3.4.6) \quad R_\alpha(f^{n(s)+N}(x))$$

$$= \sup_{i,j \geq N} \{p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+i}(x)) + p_\alpha(Tf^{n(s)+j}(x) - Sf^{n(s)+j}(x) + p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+j}(x)) + p_\alpha(Tf^{n(s)+j}(x) - Sf^{n(s)+i}(x)) + p_\alpha(Tf^{n(s)+i}(x) - Tf^{n(s)+j}(x))\}.$$

Also,

$$p_\alpha(Tf^{n(s)+i}(x) - Sf^{n(s)+i}(x))$$

$$= p_\alpha(f^{n(s)+i}(Tx) - f^{n(s)+i}(Sx))$$

$$\leq p_\alpha(f^{n(s)+i}(Tx) - f^i(Tz)) + p_\alpha(f^i(Tz) - f^i(Sz$$

$$p_\alpha(f^i(Sz) - f^{n(s)+i}(Sx))$$

$$< p_\alpha(f^i(Tz) - f^i(Sz) + \epsilon/5 = p_\alpha(Tf^i(z) - Sf^i(z) + \epsilon/5$$

follows from (3.4.5).

Using a similar computation for other terms of (3

we have for $s \geq t$,

$$R_\alpha(f^{n(s)+N}(x)) < \sup_{i,j\geq N} \{p_\alpha(Tf^i(z) - Sf^i(z)) + p_\alpha(Tf^i(z) - Sf^j(z) + p_\alpha(Tf^i(z) - Sf^j(z)) + p_\alpha(Tf^j(z) - Sf^i(z)) + p_\alpha(Tf^i(z) - Tf^j(z))\} + \epsilon$$

$$= R_\alpha(F^N(z)) + \epsilon.$$

Hence $r_\alpha(x) = \lim_N R_\alpha(f^{n(s)+N}(x)) \leq \lim_N R_\alpha(f^N(z))$

$= r_\alpha(z) + \epsilon$ for all positive inteyers N .

Since $\epsilon > 0$ is arbitrary, it follows that $r(x) \leq r_\alpha(z)$.

Hence from (3.4.4), we have

(3.4.7) $R_\alpha(z) = r_\alpha(x) \leq r_\alpha(z)$.

Supposr $R_\alpha(z) > 0$. Then since S and T have *dimjodis*, we have $r_\alpha(z) < R_\alpha(z)$, a contradiction to (3.4.7). Hence $R_\alpha(z) = 0$, proving $Sz = Tz$. Further, since $r_\alpha(x) = 0 = \lim_n r_\alpha(f^n(x))$, it follows that sequence $\{f^n(x)\}$ is a Cauchy sequence. Therefore $\lim_n f^n(x) = z$.

COROLLARY 3.5. Let $K \subseteq E$ be nonempty bounded, and let S and T be commuting nonexpansive mappings of K into K having dimjodis ; and if for some $x \in K$ a subsequence of the sequency $\{f^n(x)$, where $f = ST = TS$, has a limit z in K, then $\lim_n f^n(x) = z$ and $Sz = Tz$.

Since $K \sqsubseteq E$ is sequentially compact implies K is bounded, we have the following:

COROLLARY 3.6. Let $K \sqsubseteq E$ be nonempty sequentially compact, and let S and T be commuting continuous mappings of K into K such that

(3.6.1) S and T have *dimjodis*, and

(3.6.2) $f \doteq ST = TS$ is bounded.

Then for each x in K, some subsequence of the sequence $\{f^n(x)\}$ converges to a point z in K and $Sz = Tz$.

COROLLARY 3.7. Let $K \sqsubseteq E$ be nonempty sequentially compact, and let S and T be commuting nonexpansive mappings of K into K having *dimjodis* . Then for each x in K some subsequence of the sequence $\{f^n(x)\}$ where $f = ST = TS$, converges to a point z in K and $Sz = Tz$.

Theorm 3.4 and its corollaries generalize the corresponding results of Tan [7] and Mishra [5].

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PLANT-PARASITIC AND SOIL NEMATODES INFESTING LEGUMINOUS CROPS IN RISHIKESH AND HARIDWAR REGIONS (U.P.)

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ABSTRACT

A survey of soil samples from the roots of Leguminous crops collected from ten localities in and around Rishikesh and Haridwar (U.P.) yielded 31 species of plant-parasitic and soil nematodes. These include mainly ectoparasites, predators and free-living forms, only two endoparasitic nematodes viz. *Rotylenchulus reniformis* and *Meloidogyne javanica* were recovered. *Helicotylenchus dihystra* and *Hopolaimus indus* are the most frequent nematods found in this region.

Key words : Leguminous crops, plant-parasitic and soil nematodes.

INTRODUCTION

Leguminous crops like *Phaseolus mungo*, *P. radiata*, *P. acontifolius*, *Cajanus cajan*, *Pisum sativum* etc. are widely grown in west Uttar Pradesh. Beside serving as fodder to animals these are also important, for Nitrogen fixation. Das [1] reported several nematodes associated with leguminous crops for the first time in India. Later on many nematodes including some endoparasites viz., *Heterodera avenae*, *H. cajani*, *H. trifolli*, *Meloidogyne javanica*, *M. incognita*, *Rotylenchulus reniformis*, *Radopholus similis* have been reported infesting

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leguminous crops in several parts of India by Seshadri & Shivkumar [11], Prasad *et al.*[9], [2], Koshy [5], Koshy and Sosamma [6], Janarthanan *et al.*[3], Sitaramiah *et al.* (12), Mulk & Jairajpuri [7,8], Kaushik & Bajaj [4], Reddy [10], A world list of Nematodes pathogens associated with chickpea, ground nut, pearl millect, pigeonpea and Sorghum was provided by Sharma [13]. But Rishikesh and Haridwar region has not yet been surveyed by any worker for the recovery of nematodes associated with leguminous crops. Therefore, an attempt has been made to recover all nematode associated with leguminous crops in this region.

MATERIALS AND METHODS

The soil samples were processed by Cobb's Sieving and decantation techniques. Nematodes were isolated with the help of Baermann's funnel. The nematodes, thus recovered, were fixed with hot 4% formaline. Mounting of nematodes was done in anhydrous glycerine after slow dehydration. DeMan's formula was used for denoting body dimension. Ten localities were chosen to collect the soil samples around the roots of different leguminous crops.

RESULT AND DISCUSSION

A survey of soil samples during 1989-90 around the roots of different leguminous crops yielded as many as 31 species of plant-parasitic and soil nematodes (Table-1). All these nematodes belong to four orders of Nematoda viz., Tylenchida, Dorylaimida, Mononchida and Rhabditida. The nematodes belonging to Tylenchida are ecto-and endoparasites. Dorylaimida includes free-living as well as

ecto-parasites. Mononchids and Rhabditids are exclusively predatory and saprophagous respectively. Some nematodes have been studied in detail and are as follows:

1. *Tylenchorhynchus vulgaris* Upadhyay, Swarup and Sethi, 1972:

(5 ♀♀) L = 0.62 - 0.78 mm; a = 24-26, b = 4.1 - 4.4, c = 19-21, v = 53-55, Spear = 22-23 µm; nervering=105-110 um; excretory pore=115-128 µm.

2. *Helicotylenchus dihystra* (Cobb, 1912) Sher, 1963

(7 ♀♀) L = 0.48-0.54 mm; a = 27-30; b = 3.5-4.0; c = 35-38; v = 48-50; Spear = 23-25 µm; excretory pore = 105-145 um.

3. *Hopolaimus indicus* Sher, 1963

(10 ♀♀) L = 0.90-1.1 mm; a = 23-25, b = 4.3-4.6; c = 21-24; v = 54-57; Spear = 33-37 um; Nerve ring = 90-105 um; excretory pore = 85-98 µm.

(5 ♂♂) L = 0.82-0.90 mm; a = 21-23; b = 4.4-4.6 c=24-28; Spicules=35-40 um; Bursa = 100-115 µm.

4. *Rotylenchus* sp.

(5 ♀♀) L=0.87-0.95 mm; a=30-37; a=30-33; b=6.0-6.5; c=22-24; v=65-67; Spear=27-30 um; Nerve ring=95-110 um; excretory pore=117-120 µm.

5. *Longidorus* sp.

(5 ♀♀) L=3.6-4.5 mm; a=135-148; b=11-14; c=55-62; v=51-54; Odontostyle=73-75 µm; Odontophore=56-65 µm., Guiding ring = 23-27 µm; pre rectum = 230-245 µm, Basal Bulb=53-60 x 20-22 µm,

6. *Xiphinema* sp.

(2 ♀♀) L=1.30-1.32 mm; a=44-45; b=9-10; c=17; V=45-46; Odontostyle=65 µm; Odontophore=73 µm

7. *Actinolaimus* sp.

(10 ♀♀) L=1.75-1.89 mm; a=51-55; b=6.7-7.2; c=6-7; V=52-55; Odontostyle=18-20 µm; Odontophore= 21-24 µm.

8. *Thornenema* sp.

(5 ♀♀) L=1.30-1.36 mm; a=35-38; b=4.4-4.6; c=5.3-5.6; V=38-40; Odontostyle=11-13 µm; Odontophore= 15-20 µm.

9. *Mylonchulus minor* (Cobb, 1893) Andrassy, 1958;

(10 ♀♀) L=1.05-1.15 mm; a=23-25; b=3.2-3.5; c=30-32; V=57-58; Buccal cavity = 29-35 x 14-17 µm; tail = 34-35 µm.

10. *Iotonchus indicus* Jairajpuri, 1969;

(9 ♀♀) L=1.78-1.90 mm; a=31-35; b=4.4-4.7; c=8.1-8.6; V=57-58; Buccal cavity = 43-47 x 23-25 µm.

From the present study it is clear that large number of nematodes are associated with the leguminous crops in this region. *Helicotylenchus dihystra*, *Hoplolaimus indicus*, *Tylenchorhynchus -vulgaris*, *Eudorylaimus* sp., *Longidorus* sp. and mylonechulus minor were most abundant while *Hirschmaniella*, *Aphelenchus*, *Scutellonema*, *Paratrichodorus* and *Rotylenchulus* were least encountered here.

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TABLE-1

LIST OF NEMATODES RECORDED FROM
DIFFERENT LOCALITIES & DIFFERENT SAMPLES (Numbrs)

	NAMES OF NEMATODES	NUMBER OF SAMPLES
1.	*Longidorus* sp.	1, 2, 5, 6, 7, 9
2.	*Hoplolaimus indicus*	1, 2, 3, 4, 8, 9
3.	*Helicotylenchus dihystra*	1, 2, 3, 6, 7, 8, 9, 10
4.	*Rotylenchus* sp.	6, 8, 9
5.	*Apercelaimus* sp.	1, 2
6.	*Hirschmanniella mucronata*	1
7.	*Pratylenchus* sp.	8
8.	*Rotylenchulus reniformis* sp.	3
9.	*Eudorylaimus* sp.	1, 2, 3, 4, 5, 7, 9
10.	*Dorylaimus stagnalis*	4, 5, 6, 7
11.	*Xiphinema* sp.	3
12.	*Prodorylaimus* sp	4, 6, 9
13.	*Tylenchorhynchus vulgaris*	1, 2, 3, 4, 6, 8, 10
14.	*Thornonema* sp.	3, 4, 6, 7, 8
15.	*Scutellonema* sp.	4, 2
16.	*Mononchus agnaticus*	4, 6
17.	*Amphidelus* sp.	5, 9
18.	*Mylonchulus minor*	5, 6, 7, 8, 9, 10
19.	*Discolaimus brevis*	4, 6, 9, 10
20.	*Aphelenchus avenae*	7
21.	*Para-phelenchus* sp.	7
22.	*Paratrichodorus mirzai*	7
23.	*Iotonchus indicus*	3, 7, 8, 10
24.	*Paratylenchus* sp.	8
25.	*Meloidogyne javanica*	8, 9
26.	*Aphelenchoides andrassyi*	8
27.	*Proleptonchus* sp.	9
28.	*Actinolaimus* sp.	2, 3, 5, 7, 8
29.	*Chronogaster* sp.	2, 3
30.	*Plectus* sp.	4
31.	*Rhabditis* sp.	8

Table-1 Contd.

LOCALITIES : SOIL AROUND THE ROOTS OF :

1. Pigeon Pea (*Cajanus cajan*) from Ram Nagar, Rishikesh, Dehradun.

2. Pigeon Pea (*Cajanus cajan*) from Geeta Nagar, Rishikesh, Dehradun.

3. Dhaicha (*Crotoloria junicea* L.) from Bhadra Bad, Haridwar.

4. Urd (*Phaseolus radiata*) from Sikenderpur, Haridwar.

5. Pigeon Pea (*Cajanus cajan*) from Dhanorie, Haridwar.

6. Bean Plant (*Dolichos lab-lab*) from Shyampur, Haridwar.

7. Chick Pea (*Cicer aretenum*) or Gram from Ganga Nagar, Rishikesh, Dehradun.

8. Matar (*Pisum sativum*) from I.D.P.L., Virbhadra, Rishikesh.

9. Chick Pea (*Cicer aretenum*) or Gram from Gomani Wala, Shyampur, Rishikesh, Dehradun.

10. Master (*Pisum sativum*) from Dhalwala, Rishikesh, Dehradun.

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बहुमानी संकारकों के अनुक्रमों का अभिसरण

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(प्राप्त 6.12.1990)

सारांश

इस प्रपत्र में दो बहुमानी प्रतिचित्रणों के लिए इसिकावा पुनरावृत्तिक विधि के अधीन दो स्थिर बिन्दु प्रमेय स्थापित किये गये हैं, जो नैम्पल्ली-सिंह एवं सिंह व अन्य के परिणामों को विस्तारित करते हैं।

AMS (MOS) Subject Classifications (1980, Revision 1985). 54H25, 47H10.

ABSTRACT

CONVERGENCE OF SEQUENCES OF MULTIVALUED OPERATORS

S.L. Singh*, U.C. Guirola* and S.N. Mishra**

The notion of Ishikawa iterates is extended to a pair of multivalued maps and it is shown hat under certain contractive conditions the limit of the sequence of Ishikawa iterates, when it converges, is a common fixed point of the maps.

भूमिका एवं प्रारंभिकी

रोअडेस् [10], हिक्स-कुबिसेक [3], तथा अन्य कई गणितज्ञों (देखें, उदाहरणार्थ [9], [10], [12]) ने दिखाया है कि कोई प्रतिचित्रण T जो किसी विशेष संकुचित शर्त को सन्तुष्ट करता है उसके स्थिर बिन्दु पर वह मान पुनरावृत्तिक (देखें [2], [14]) अनुक्रम अभिसरित होता है जो स्वयं अभिसारी हो, दूसरी ओर नैम्पल्ली - सिंह [9] ने व्यापक संकुचन शर्त के साथ यह देखा कि कोई अभिसारी इशिकावा [4] पुनरावृत्तिक अनुक्रम उसी बिन्दु पर अभिसरित होता है यदि वह बिन्दु प्रतिचित्रण T का स्थिर बिन्दु हो जब-कि नायडू-प्रसाद [8] ने [9] में प्रदत्त कुछ परिणामों का अध्ययन इशिकावा पुनरावृत्तिक अनुक्रम के अधीन एक युग्म प्रतिचित्रण के लिये किया हाल ही में कुहफिटिंग [6] ने मान पुनरावृत्तिक विधि हेतु क्रेस्नोसेस्कोज [5] की पुनरावृत्तिक विधि का विस्तार बहुमानी प्रतिचित्रणों के लिये किया इन सभी धारणाओं से प्रेरणा लेकर सिंह [15] ने बहुमानी प्रतिचित्रणों के लिये इशिकावा पुनरावृत्तिक विधि को पारिभषित किया तथा इसके लिये स्थिर बिन्दुओं का सन्निकटन प्राप्त किया।

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प्रस्तुत प्रपत्र में एक युग्म बहुमानी प्रतिचित्रणों के लिए इशिकावा पुनरावृत्तिक विधि के अधीन दो प्रमेयों को सिद्ध कर रहे हैं. प्राप्त परिणाम सिंह [15] का व्यापकीकरण करते हैं. साथ ही रोअडेस् [10], हिक्स - कुबिसेक [3], नैम्पल्ली - सिंह [9], नायडू-प्रसाद [8] तथा कुहाफिटिग [6] द्वारा एक मानी एवं बहुमानी प्रतिचित्रणों के लिए स्थापित परिणाम विशिष्ट दशाओं के साथ समुचित रूप से प्राप्त किये जा सकते हैं.

सामान्यतया नाडलर [7] एवं सिंह [15] में प्रयुक्त संकेताक्षरों का अनुसरण किया गया है.

मान लें $(X, \|\ \|)$ एक मानकित रैखिक समष्टि है, $CL(X)$ समष्टि X के अरिक्त संवृत उपसमुच्चयों का संग्रह है तथा $(CL(X), H)$ व्यापकीकृत हाउसडार्फ दूरीक समष्टि है.

निम्न प्रमेयिका सुज्ञात है (अधिक जानकारी के लिए देखें रूस [13]).

प्रमेयिका. मान लें $q > 1$ एवं $A, B \in CL(X)$. तब A के प्रत्येक अवयव x के लिए B में एक बिन्दु y का अस्तित्व इस प्रकार प्राप्त होता है कि

$$d(x,y) \leq qH(A,B).$$

हम X के संवृत अवमुख उपसमुच्चय C से $CL(C)$ पर पारिभाषित प्रतिचित्रणों S, T के लिए [15] में चर्चित निम्न पुनरावृत्तिक विधि का अनुसरण करेंगें.

(1) $x_0 \in C$;

(2) $y_n = b_n p_n + (1-b_n) x_n, n \geq 0, p_n \in Sx_n$,

(3) $x_{n+1} = (1-a_n) x_n + a_n q_n, n \geq 0$;

उक्त प्रमेयिका के अनुसार $q_n \in Ty_n$ इस प्रकार है कि

(4) $\|p_n - q_n\| = k^{-h} H(Sx_n, Ty_n)$, जहां $h, k \in (0,1)$;

(5) सभी n के लिए $0 \leq a_n, b_n \leq 1$;

(6) सीमा $a_n \geq 0$.

ध्यान देने योग्य है कि यदि $T: C \to C$ तो शर्त (4) आसानी से सन्तुष्ट हो जाती है क्योंकि $k^{-h} > 1$.

परिणाम

प्रमेय 1. मान लें C मानकित रैखिक समष्टि X का संवृत अवमुख उपसमुच्चय है तथा $S, T: C \to CL(C)$. पुनः मान लें कि पूर्व पारिभाषित अनुक्रम $\{x_n\}$ बिन्दु z पर अभिसरित होता है. यदि C के सभी अवयवों x, y के लिए S, T शर्त

(7) $H(Sx, Ty) \leq$

अधिकतम $\{ ||x-y||, D(x,Sx), D(y,Ty), [D(y,Sx)+D(x,Ty)] \}$

को संतुष्ट करे, तब $z \in Sz \cap Tz$.

उपपत्ति . क्योंकि $x_n \to z$ और $\{a_n\}$ शून्य से उपरि परिबद्ध है, अतः

$$||x_{n+1}-x_n|| = a_n||x_n-q_n||$$

एवं

$$||q_n-z|| \leq ||z-x_n|| + ||x_n-q_n||$$

अर्थात् $||x_n-q_n||$ एवं $||q_n-z||$ शून्य पर अभिसरित होते हैं. पुनः त्रिभुज असमिका एवं (2) से

$$||x_n-y_n|| = b_n||x_n-p_n|| \leq ||x_n-p_n||$$

$$||x_n-p_n|| \leq ||x_n-q_n|| + ||q_n-p_n||$$

$$||y_n-q_n|| \leq ||y_n-x_n|| + ||x_n-q_n||$$

$$\leq ||x_n-p_n|| + ||x_n-q_n||$$

$$\leq ||x_n-q_n|| + ||p_n-q_n|| + ||x_n-q_n||$$

$$\leq 2||x_n-q_n|| + ||p_n-q_n||$$

एवं

$$||y_n-p_n|| = (1-b_n) ||x_n-p_n|| \leq ||x_n-p_n||.$$

अतः

$$||p_n-q_n|| \leq k^{-h}H(Sx_n,Ty_n)$$

$$\leq k^{-h} \text{ अधिकतम } \{ ||x_n-y_n||, D(x_n,Sx_n), D(y_n,Ty_n), [D(x_n,Ty_n)+D(y_n,Sx_n)]\}$$

$\le k'$ अधिकतम $\{ ||x_n-p_n||, ||y_n-q_n||, ||x_n-q_n|| + ||y_n-p_n||\}$,

जहाँ $k^{1-h} = k'$. इसलिए

$$||p_n-q_n|| \le \frac{2k'}{1-k'} ||x_n-q_n||.$$

इस प्रकार $n \to \infty$ लेने पर

$$||p_n-q_n|| \to 0, \ ||x_n-p_n|| \to 0,$$

तथा

$$||p_n-z|| \to 0, \ ||y_n-q_n|| \to 0$$

एवं (2) से $||y_n-z|| \to 0$.

शर्त (7) द्वारा,

$D(z,sz)$

$\le ||z-q_n||+D(q_n,Sz)$

$\le ||z-q_n||+H(Ty_n,Sz)$

$\le ||z-q_n||+k$ अधिकतम$\{ ||y_n-z||, D(y_n,Ty_n), D(z,Sz), [D(y_n,Sz)+D(z,Ty_n)\}$

$\le ||z-q_n||+k$ अधिकतम$\{ ||y_n-z||, ||y_n-q_n||, D(z,Sz), ||y_n-z|| + D(z,Sz) + ||z-q_n||]\}$.

स्पष्टतः $n \to 0$ लेने पर, $z \in Sz$. इसी तरह $z \in Tz$.

अब प्रतिचित्रणों $S,T:C \to CL(C)$ के लिए निम्न शर्तों पर विचार करें।

(8a) $H(x,Sx)+H(y,Ty) \leq a||x-y||,\ 1 \leq a < 2;$

(8b) $H(x,Sx) + H(y,Ty)$

$\leq b[D(x,Ty) + D(y,Sx) + ||x-y||]$, जहाँ $1/2 \leq b < 2/3,$

(8c) $H(x,Sx) + H(y,Ty) + H(Sx,Ty)$

$\leq c[D(x,Ty) + D(y,Sx)],\ 1 \leq c < 3/2,$

(8d) $H(Sx,Ty) \leq k$ अधिकतम $\{ ||x-y||, D(x,Sx),$

$D(y,Ty), 1/2[D(x,Ty) + D(y,Sx))\},\ 0 \leq k < 1$

प्रमेय 2. मान लें C मानकित रैखिक समष्टि X का एक संवृत अवमुख उपसमुच्चय है एवं पूर्व पारिभाषित अनुक्रम $\{x_n\}$ किसी बिन्दु $z \in C$ पर अभिसरित होता है. यदि C के प्रत्येक अवयव x,y के लिए प्रतिचित्रण S,T उक्त शर्तों (8a)-(8d) में से कम से कम एक शर्त को सन्तुष्ट करे तथा (8e) भी सन्तुष्ट हो तब $z \in S_z \cap T_z$, जहाँ

(8e) $k^{-h} \leq 2$ या सीमा $b_n > \dfrac{(c-1-k^h)}{(c+1)}$.

उपपत्ति. हम किरिक [1] के निम्न सरल कथन का कई बार प्रयोग करेंगें,

किसी $x \in C$ एवं $y \in B$ के लिए

$$||x-y|| = H(x,B),\ B \in CL(C).$$

चूँकि $p_n \in Sx_n$ तथा $q_n \in Ty_n$,

$$2||p_n-q_n||$$

$$\leq (||x_n-p_n|| + ||y_n-q_n||) + ||x_n-q_n|| + ||y_n-p_n||$$

$$\leq H(x_n,Sx_n) + H(y_n,Ty_n) + ||x_n-q_n|| + ||y_n-p_n|| .$$

इनसे जैसा कि प्रमेय 1 की उपपत्ति में देखा गया, निम्न सरलता से प्राप्त होते हैं :

$$||x_n-q_n|| \to 0 \qquad ||q_n-z|| \to 0.$$

यदि $x = x_n$ एवं $y=y_n$ के लिए (8a) सन्तुष्ट हो तब

$$2\,||p_n-q_n||$$

$$\le a||x_n-y_n|| + ||x_n-q_n| + ||y_n-p_n||$$

$$\le ab_n(||x_n-q_n|| + ||q_n-p_n||) + ||x_n-q_n||$$

$$+ (1-b_n)(||x_n-q_n|| + ||q_n-p_n||)$$

अर्थात्

$$||p_n-q_n||$$

$$\le t||x_n-q_n||, \text{ जहाँ } t = (2 + (a-1)\, b_n)/(1-(a-1)b_n).$$

यदि $x = x_n$ एवं $y = y_n$ के लिए (8b) सन्तुष्ट हो, तब

$$2||p_n-q_n||$$

$$\le b(D(x_n,Ty_n) + D(y_n,Sx_n) + ||x_n-y_n|| + (||x_n-q_n||$$

$$+ ||y_n-p_n||)$$

$$\le (b+1)(||x_n-q_n|| + ||y_n-p_n||) + b||x_n-y_n||$$

अर्थात्

$$||p_n-q_n||$$

$$\le t_2\, ||x_n-q_n||, \qquad t_2 = (2 + 2b - b_n)/(1-b+b_n).$$

यदि $x = x_n$ एवं $y = y_n$ के लिए (8c) सन्तुष्ट हो, तब

$||p_n - q_n|| \leq t\, H(Sx_n, Ty_n)$, जहाँ $t = k^{-h}$.

अतः

$$(2t + 1)\, ||p_n - q_n|| = t\, ||p_n - q_n|| + t\, ||p_n - q_n||$$

$$+ ||p_n - q_n|| \leq t(||p_n - y_n|| + ||y_n - q_n||)$$

$$+ t(||x_n - p_n|| + ||x_n - q_n|| + tH(Sx_n, Ty_n)$$

$$\leq t\, H(y_n, Ty_n) + tH(x_n, Sx_n) + tH(Sx_n, Ty_n)$$

$$+ t(||x_n - q_n|| + ||y_n - p_n||)$$

$$\leq ct\, D(x_n, Ty_n) + D(y_n, Sx_n) + t\, (||x_n - q_n|| + ||y_n - p_n||)$$

$$\leq t(c+1)\, (||x_n - q_n|| + ||y_n - p_n||)$$

अथवा

$$||p_n - q_n|| \leq t_3 ||x_n - q_n||,$$

जहाँ $t_3 = t(c+1)\, (2 - b_n)/(1 + t(1 - c(1 - b_n) + b_n))$.

अन्ततः यदि $x = x_n$ एवं $y = y_n$ के लिए (8d) सन्तुष्ट हो, तब

$$||p_n - q_n|| \leq$$

$$tk \text{ अधिकतम } \{||x_n - y_n||,\ D(x_n, Sx_n),\ D(y_n, Ty_n),$$
$$1/2[D(x_n, Ty_n) + D(y_n, Sx_n)]\}$$

$$\leq tk(2||x_n - q_n|| + ||q_n - p_n||).$$

अस्तु,

$||p_n-q_n||$

$\leq t_4||x_n-q_n||$, जहाँ $t_4 = 2tk/(1-tk)$.

इसलिए $x = x_n$ एवं $y = y_n$ के लिए

(9) $||p_n-q_n|| \leq$

अधिकतम $\{t_1, t_2, t_3, t_4\} ||x_n-q_n||$.

स्पष्टतः किसी b_n [0,1] के लिए $t_4 > 0$ एवं t_1, t_2 धनात्मक हैं.

अब t_3 को दृष्टिगत रखते हुए किसी $b_n \in [0, 7]$ के लिए t_3 धनात्मक होगा यदि

$1+t(1-c\ (1-b_n)+b_n) > 0$,

अर्थात् $1+t(1-3(1-b_n)/2+b_n) > 0$ (क्योंकि c व 3/2 आपस में काफी निकट हो सकते हैं).

इसलिए $i+t(1-3/2) > 0$ या $t < 2$.

स्पष्टतया किसी $b_n \in [0,1]$ के लिए $t=2$ होने पर t_3 धनात्मक होगा.
यदि $t > 2$ तब t_3 धनात्मक होगा, यदि

$1+t(1-c(1-b_n)+b_n) > 0$,

अर्थात यदि

$b_n > (c-1)/(t-1)/(tc+t) = (c-1-1/t)/(c+1)$.

इसलिए (8e) को मद्देनज़र रखते हुए t_3 धनात्मक होगा. इस प्रकार (9) से

$||p_n-q_n|| \to 0, \quad ||x_n - P_n|| \to 0$

एवं $||p_n - z|| \to 0$, जब $n \to \infty$ लें.

अब हम यह दिखायेंगे कि S का स्थिर बिन्दु z है. $x = z$, $y = y_n$ के लिए प्रतिबन्धों (8a)-(8a) का परीक्षण करेंगें. सर्वप्रथम (8a) लेकर

(10a) $H(z, Sz) + H(y_n, Ty_n)$

$\leq a||y_n - z||.$

इसी तरह (8b) से

$||y_n - q_n|| + H(z,Sz)$

$\leq b(D(y_n,Sz) + D(z,Ty_n) + ||y_n - z||)$

$\leq b(||y_n - z|| + D(z,Sz) + ||z-q_n|| + ||y_n - z||)$

$\leq b(2||y_n - z|| + ||z-q_n|| + H(z,Sz))$

अर्थात्

(10b) $(1-b)\ H(z,Sz)$

$\leq 2b||y_n - z|| - (1-b)||z-q_n||.$

पुन: (8c) से

$2H(z,Sz)$

$\leq H(z,Sz) + ||z-y_n|| + H(y_n,Ty_n) + H(Ty_n,Sz)$

$\leq ||z-y_n|| + c(D(y_n,Sz) + D(z,Ty_n))$

$\leq ||z-y_n|| + c(||y_n - z|| + D(z,Sz) + ||z-q_n||)$

$\leq (c+1)\ ||z-y_n|| + cH(z,Sz) + c||z-q_n||$

अर्थात्

(10c) $(2-c)\ H(z,Sz)$

$$\leq (c+1)\ ||z-y_n|| + c||z-q_n||.$$

अतः (8d) से

$$D(z, Sz) \leq ||z-q_n|| + H(Ty_n, Sz)$$

$$\leq ||z-q_n|| + k \text{ अधिकतम } \{||y_n-z||,\ D(y_n, Ty_n),$$
$$D(z,Sz),\ 1/2(D(y_n,z) + D(z,Ty_n))\}$$

(10d) $\leq ||z-q_n|| + k$ अधिकतम $\{\ ||y_n-z||,$
$$||y_n-q_n||,\ D(z,\ Sz),\ 1/2(||y_n-z|| + D\ (z,Sz) + ||z-q_n||)\}.$$

अतः (10a) - (10c) में सीमा $n \to \infty$ लेने पर

$H(z,Sz) = 0$ एवं (10d) से $D(z,Sz) = 0$ अर्थात् $z \in Sz$.

इसी प्रकार हम सिद्ध कर सकते हैं कि $z \in Tz$.

टिप्पणी 1. प्रमेय 1-2 में $S = T$ लेने पर सिंह [15] के परिणाम उपप्रमेय के रूप में प्राप्त हो जाते हैं.

टिप्पणी 2. प्रमेय 1 में $S=T: C \to C$ लेने पर नैम्पल्ली-सिंह [9] की प्रमेय 1.2 को उपप्रमेय के रूप में प्राप्त किया जा सकता है.

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फार्म—४

प्राकृतिक एवं भौतिकीय विज्ञान शोध पत्रिका
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मैं, भारत भूषण, कुलसचिव, गुरुकुल कांगड़ी विश्वविद्यालय, हरिद्वार, घोषित करता हूं कि उपरिलिखित तथ्य मेरी जानकारी के अनुसार सही हैं।

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